

THE
ELEMENTS
OF
TRIGONOMETRY.

CONTAINING,

The Properties, Relations, and Calculations of
SINES, TANGENTS, SECANTS, &c.

The Doctrine of the SPHERE, and the PRIN-
CIPLES of PLAIN and SPHERICAL
TRIGONOMETRY.

All plainly and clearly demonstrated.

By William Emerson.

*Quicquid in Astronomicis, Geographicis, vel Nauticis efficiendum,
Trigonometriæ attribuendum est.*



L O N D O N :

Printed for W. INNYS, in Pater-noster Row. 1749.

THE

ELEMENTS

OF

TRIGONOMETRY

CONTAINING

The Properties of the Sines, Tangents, Secants, &c.
The Rules for the Resolution of Triangles
The Logarithmic Trigonometrical Tables
The Logarithmic Trigonometrical Tables

All plain and clearly demonstrated.

By JOHN WALLIS, M.A. Fellow of the Royal Society.
The second Edition, with Additions.



LONDON:

Printed for W. Bland in Pall-mall, 1719.

THE P R E F A C E.

THE Use of Trigonometry is so great in all the Parts of the Mathematics, that He must have made a very little Progress therein, who is not sensible of it. Its help is call'd in upon every Occasion, and its great Service is clearly apparent in Calculations of all Sorts, both upon the Earth, upon the Seas, and in the Heavens. By this, the Distances of Objects upon the Earth may be certainly known, if they can but be seen; tho' we cannot come near to measure them: Likewise the geographical Distances of Places on the Earth; and their several Positions to one another. Navigation depends entirely upon it. Surveying and Dialling owe their greatest Exactness to it. It is of singular Service in military Affairs: And Mars, without this, might live peaceably at Home. Upon the Wings of Trigonometry (as Plato says), we mount up from the Earth to the Heavens, measure the Distances of all the Stars, and range them in their proper Order. And without it Urania's Sons may throw aside their Instruments, their Books and Tables; or rather, without this, they would never have had any such thing; and consequently Mankind had remained utterly ignorant of this most beautiful System of the World. In short, the Art of Trigonometry is of such universal Use and Extent, that it would be an endless Task to enumerate all the various purposes to which it is subservient; and the most important Branches of Knowledge would be lost and useless if we wanted it.

This makes me wonder that no body has given us an entire System of this Art ; which shall contain all the Principles, as well as the Rules of Practice. Instead of that, most Authors content themselves with little more than explaining the common Cases of solving Triangles, and fill up their Books with heaps of Examples, to shew the Use and Application thereof : Whilst they dip very little into the Theory, and leave the Principles thereof very much in the Dark.

In this Treatise I have adventured to lay down the Whole both Theory and Practice ; and to take in all Things of any Consequence that any way belong to the Subject. And here I account all these Properties of Triangles or other Figures that have any Relation to the measuring their Angles by Degrees, or by Sines, and Tangents, &c. to belong to Trigonometry, as their proper Subject : referring their other Properties to Geometry, which have no such Regard to the Measures of their Angles.

In pursuance of this Design, I have in the first Book laid down in a few Propositions, the Relations of Sines, Cosines, &c. of Arches ; likewise of the double Arches, and of the Sums and Differences of Arches. Among which there are a few Propositions, so extensive and general, as to comprehend a great Number of particular Propositions, evident only by Inspection.

Then I have given the Method of calculating natural and artificial Sines, Tangents, &c. of any Arch : And from thence those of their multiple Arches. And here I have been obliged to have Recourse to the Method of Fluxions, and the Method of Increments ; there being no other possible way to effect it. Which if any of my Readers happen not to understand, they may pass by the Investigations, and make equal Use of the Conclusions, nevertheless.

Then

The PREFACE.

Then you have a few general Propositions concerning angular Sections, Inscription of Polygons, and the Properties of Chords inscrib'd in a Circle.

In the second Book you have plain Trigonometry with the Solution of all the Cases, several ways. Right angled Triangles are resolv'd arithmetically, logarithmically, and algebraically: And oblique ones logarithmically and algebraically. By the first Method all the Cases of right angled Triangles are resolved by common Arithmetic, without any Books or Tables whatever. And this comes very near the Truth, and is sufficiently exact for common Uses: And is no further burthensome to the Memory than getting by heart two or three Proportions, with certain fixt Numbers; and the 47. I. Euclid. And the same way may oblique Triangles be resolved, by letting fall a Perpendicular.

In the third Book you have all that is material in the Doctrine of the Sphere; likewise the Properties of Spherical Triangles; and the Principles of Spherical Trigonometry; in which are some compound or general Propositions, including several particular ones. Among these there are some that I have taken the Liberty to demonstrate algebraically, to avoid a more prolix Method of Demonstration. Then follows the Solution of all the Cases of spherical Triangles; in oblique ones both logarithmically and algebraically.

In the first Sect. of the first Book, I have inserted several Scholia here and there: The use of which to Algebraists, will be too evident to be further insisted on. And for the same Reason it was, that I inserted the Solution of the Cases algebraically.

All these Things being laid down here in this Book, in the shortest Manner possible; I think I may venture to entitle it The Elements of Trigonometry. I won't say

The PREFACE.

say that I have quite exhausted the Subject, but I am of Opinion that I have omitted little or nothing of any Consequence.

It would be needless giving any Examples in Numbers, as likewise the Application of Trigonometry to any Part of Mathematicks: That being very easy for the Reader to do himself; especially as all Authors abound in these Things.

If what I have done meet with general Approbation, I have my End; if not I can only say,

In magnis voluisse sat est.

W. Emerson.



THE

THE

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BOOK I.

Characters used in this Book.

+ Addition.

— Subtraction of the following Quantity.

× Multiply'd by.

⊥ Perpendicular to.

∥ Parallel to.

:: Proportion.

> Greater than.

< Lesser than

∠ Angle.

Δ The Difference.

S. Sine; Cos. Cosine; T. Tangent; Cot. Cotangent;
Sec. Secant; Cofec. Cofecant; vers. versed sine;
Sup. Supplement; Seg. Segments; diff. difference.

THE
ELEMENTS
OF
TRIGONOMETRY.

BOOK I.

The Properties of Sines, Tangents, Secants, &c. of Arches; the Calculation of them, both natural, and artificial; and also those of multiple Arches; angular Sections, &c.

DEFINITIONS.

1. **A**N *Arch* of a Circle is any Part of its Circumference as A B. A Quarter of the Circumference is called a *Quadrant* as A D; and the half a *Semi-circle* as A D I. FIG. 1.

2. The *Radius* is a Line drawn from the Center to the Circumference, as C A.

3. The *Complement* of an Arch is what it wants of a Quadrant, B D is the Complement of A B.

4. The *Supplement* is what it wants of a Semi-circle, as B I is the Supplement of A B.

Cor. 1. The Difference of two Arches is = Difference of their Supplements.

Cor. 2. The Complement of half an Arch is equal to half the Supplement of the whole Arch. For let Q be a Quadrant and A an Arch; then $Q - \frac{1}{2}A$ is the Complement of half the Arch, and $2Q - A$ is the

B the

FIG. 1. the Supplement of the Whole, and $Q - \frac{1}{2}A$ is half the Supplement of the Whole.

Cor. 3. The Difference of an Arch and its Complement is equal to the Complement of twice the Arch.

$A \oslash Q - A$ is the Difference of the Arch and its Complement, and $2A \oslash Q$ is the Complement of twice the Arch, but $A \oslash Q - A = 2A \oslash Q$.

5. The *Chord* or *Subtense* of an Arch AB, is a right Line AB drawn between the extrem Points of the Arch.

6. The *Sine* (or right Sine) of an Arch AB, is a right Line BF drawn from one End of the Arch perpendicular to the Diameter passing thro' the other End of it.

Cor. The Sine of an Arch is half the Cord of the double Arch.

7. The *Cofine* of an Arch is the Distance between the Center and the Sine, as CF.

8. The *versed Sine* of an Arch AB, is the Part of the Diameter, between the Beginning of the Arch and the Sine, as AF.

Cor. The *versed Sine* + *Cofine* = Radius.

9. The *Coverfed Sine* is the Line DE, between the End of the Quadrant AD, and BE perpendicular to DC.

10. The *Tangent* of an Arch BA is the Line AG drawn from one End A of the Arch perpendicular to the Diameter, till it cuts the Line CBG drawn thro' the other End of the Arch.

11. The *Cotangent* of an Arch AB is the Line DH drawn perpendicular to DC at the End of the Quadrant AD, till it meet the Line CBH drawn thro' the Top of the Arch AB.

12. The *Semi-tangent* of an Arch AB, is the Line KC intercepted between the Center and the Line IK passing thro' the End B of the Arch.

Cor. The *Semi-tangent* of an Arch is = tangent of half the Arch. For if $AT = TB$, then $\angle ACT = \angle AIB$. Whence $CR = AR$.

13. The

13. The *Secant* of an Arch AB is the Line CG FIG. 1.
drawn from the Center of the Circle, thro' the Top
of the Arch AB , till it meets the Tangent AG .

14. The *Cofecant* of an Arch AB is the Line CH
drawn from the Center, thro' the End B of the Arch,
till it meets the Co-tangent DH in H .

Cor. 1. Hence the Cofine, Cotangent, Cofecant,
covered Sine of an Arch, is equal to the Sine, Tan-
gent, Secant, versed Sine, of its Complement re-
spectively.

For the Sine EB of the Arch DB (the Comple-
ment of AB) is $= CF$ the Cofine of AB . Also DH
the Cotangent of AB , is the Tangent of its Com-
plement DB . Also CH the Cofecant of $AB =$ Se-
cant of its Complement DB . And lastly DE the
covered Sine of AB is the versed Sine of its Com-
plement DB .

Cor. 2. A Sine, Tangent and Secant, Cofine, Co-
tangent, Cofecant, and covered Sine are common
to two Arches which are the Supplements of each
other. For these Lines belong as well to the Arch
 IB , as to AB .

15. A *Degree* is the 360th Part of the Circumfe-
rence of every Circle; and therefore a Semi-circle
contains 180 Degrees, and a Quadrant 90°. Also
the 60th Part of a Degree is called a Minute, the 60th
Part of a Minute a Second, &c.

16. An *Angle* ACB is the Inclination of two Lines
 AC , CB ; and the Measure of that Angle is the
Degrees in the Arch AB .

Note, The middle Letter stands at the Angle.

SCHOLIUM.

Tho' the Cofine, Tangent, Cotangent and Secant
are common to two Arches which are the Supple-
ments to each other; yet in an Arch greater than a
Quadrant, these Lines are truly negative, and must
be so esteemed, because they are drawn the contrary
way to those in an Arch less than a Quadrant. And

FIG. therefore the Sine, covered Sine, and Cofecant, are
 1. always affirmative in all Arches less than a Semi-circle, but negative in greater Arches. And in general the Sines of all Arches in the first and second Quadrants are affirmative, in the third and fourth negative. The Cosines in the first and fourth Quadrants are affirmative, in the second and third negative. The Tangents in the first and third Quadrants are affirmative, in the second and fourth negative. The Cotangents in the first and third are affirmative, in the second and fourth negative. The Secants in the first and fourth are affirmative, in the second and third negative. The Cofecants in the first and second are affirmative, in the third and fourth are negative. All versed Sines are affirmative: Moreover Sines, Tangents, &c. of negative Arches are contrary in their Signs to those of affirmative Arches.



SECT

S E C T. I.

The Relation of the Sines, Tangents and Secants of Arches.

P R O P. I.

In any Arch, these are respectively porportional,

Radius : Sine : Cosine ::

Secant : Tangent : Radius ::

Cofecant : Radius : Cotangent.

The Meaning of the Proposition is, that any four Quantities lying in Form of a Rectangle are proportional. As thus; Rad : Sine :: Secant : Tangent. Or Rad : Cosine :: Secant : Radius. Or Tangent : Radius :: Radius :: Co-tangent. Or Cotangent : Cosine :: Cofecant : Radius. And so of others. The same is to be understood of the following Propositions. Likewise if you have several Quantities placed thus, $A : B : C :: D : E : F$, the meaning is, that $A : B :: D : E$, or $A : C :: D : F$, or $B : C :: E : F$, &c. or lastly if $A : B :: C : D :: E : F$, &c. it signifies that $A : B :: C : D$, or $A : B :: E : F$, or $C : D : E : F$, and so of others.

For in the Similar-triangles BFC, GAC, CDH; $BC : BF : CF :: GC : GA : AC :: CH : CD : DH$.

Cor. 1. Radius Square = Sine Square + Cosine Square = Secant Square — Tangent Square = Cofecant Square — Cotangent Square. For the Triangles BFC, GAC; CDH are right angled.

Cor. 2. Radius Square = Tangent $\times$ Cotangent = Cosine $\times$ Secant = Sine $\times$ Cofecant.

Cor. 3. In any Arches, the Sine is as the Cosine $\times$ Tangent, or as Tangent divided by Secant, or as Cosine divided by Cotangent, or reciprocally as the Cofecant.

Cor. 4. Tangent is as the Sine $\times$ Secant, or as
B 3 Sine

FIG. 1. $\frac{\text{Sine}}{\text{Cofine}}$, or as $\frac{\text{Secant}}{\text{Cofecant}}$, or reciprocally as the Cotangent.

Cor. 5. Secant is as Tangent $\times$ Cofecant, or as $\frac{\text{Tangent}}{\text{Sine}}$, or as $\frac{\text{Cofecant}}{\text{Cotangent}}$, or reciprocally as the Cofine.

Cor. 6. Cofine is reciprocally as the Secant, Cotangent reciprocally as the Tangent, Cofecant reciprocally as the Sine.

Cor. 7. The Sum of the Squares of the Sine and versed Sine of an Arch = 4 times the Square of the Sine of half the Arch. For $AB = 2AL$, and the Triangle ABF is right angled.

Cor. 8. The Cord of an Arch is a mean Proportional between the versed Sine and Diameter. For AF, AB, AI are continually Proportional.

SCHOLIUM.

Let A = any Arch. t = Tangent σ = Cofecant.
 r = Radius. τ = Cotangent v = versed Sine.
 s = Sine. f = Secant V = vers. S. Sup.
 c = Cofine.

Then

$$\text{I. } s = \sqrt{rr - cc} = \frac{ct}{r} = \frac{tr}{f} = \frac{tr}{\sqrt{rr + tt}} = \frac{rc}{\tau} = \frac{rr}{\sqrt{rr + \tau\tau}} \\ = \frac{rr}{\sigma} = \frac{fc}{\sigma} = \frac{t\tau}{\sigma} = \frac{r}{f} \sqrt{ff - rr} = \sqrt{Vv} = \sqrt{2rv - vv}.$$

$$\text{II. } c = r - v = V - r = \sqrt{rr - ss} = \frac{rs}{t} = \frac{rr}{\sqrt{rr + tt}} = \frac{rr}{f} \\ = \frac{t\tau}{f} = \frac{s\sigma}{f} = \frac{s\tau}{r} = \frac{r\tau}{\sigma} = \frac{r\tau}{\sqrt{rr + \tau\tau}} = \frac{r}{\sigma} \sqrt{\sigma^2 - rr}.$$

$$\text{III. } t = \frac{rs}{c} = \frac{rs}{\sqrt{rr - ss}} = \frac{r}{c} \sqrt{rr - cc} = \frac{rr}{\tau} = \frac{rr}{\sqrt{\sigma^2 - r^2}} \\ \frac{cf}{\tau} = \sqrt{f^2 - rr} = \frac{s\sigma}{\tau} = \frac{r\sqrt{2rv - vv}}{r - v} = \frac{r\sqrt{2rV - V^2}}{V - r}.$$

IV. $\tau =$

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FIG.
1.

$$\text{IV. } \tau = \sqrt{\sigma\sigma - rr} = \frac{rr}{t} = \frac{s\sigma}{t} = \frac{cf}{t} = \frac{rc}{s} = \frac{r}{s} \sqrt{rr - ss} =$$

$$\frac{rc}{\sqrt{rr - cc}} = \frac{rr}{\sqrt{ff - rr}} = \frac{r - v}{\sqrt{2rv - vv}} = \frac{V - r}{\sqrt{2rV - V^2}} =$$

$$\text{V. } f = \sqrt{rr + tt} = \frac{rr}{c} = \frac{rt}{s} = \frac{r^3}{s\tau} = \frac{r\sigma}{\tau} = \frac{t\tau}{c} = \frac{s\sigma}{c} =$$

$$\frac{rr}{\sqrt{rr - ss}} = \frac{r}{\tau} \sqrt{rr + \tau\tau} = \frac{r\sigma}{\sqrt{\sigma\sigma - rr}} = \frac{rr}{r - v} = \frac{rr}{V - r} =$$

$$\text{VI. } \sigma = \sqrt{rr + \tau\tau} = \frac{rr}{s} = \frac{rf}{t} = \frac{r\tau}{c} = \frac{r^3}{ct} = \frac{t\tau}{s} = \frac{cf}{s} =$$

$$\frac{rr}{\sqrt{rr - cc}} = \frac{r}{t} \sqrt{rr + tt} = \frac{rf}{\sqrt{ff - rr}} = \frac{rr}{\sqrt{2rv - vv}} =$$

$$\text{VII. } v = 2r - V = r - c = r - \sqrt{rr - ss} = r - \frac{rr}{\sqrt{rr + tt}} =$$

$$r - \frac{r\tau}{\sqrt{rr + \tau\tau}} = r - \frac{rr}{f} = r - \frac{r}{\sigma} \sqrt{\sigma\sigma - rr} =$$

$$\text{VIII. } V = 2r - v = r + \sqrt{rr - ss} = r + c = r + \frac{rr}{\sqrt{rr + tt}} =$$

$$r + \frac{r\tau}{\sqrt{r^2 + \tau^2}} = r + \frac{r}{\sigma} \sqrt{\sigma^2 - rr} = \frac{s+t}{t} r = r + \frac{s\tau}{r} = r + \frac{rr}{f} =$$

$$\frac{\tau + \sigma}{\sigma} r.$$

The covered Sine is easily found express'd by any of the rest, by only subtracting the Sine from Radius.

PROP. II.

In any Arch these Lines are respectively proportional.

Radius : Sine of an Arch : Cosine Arch ::
 2 Sine Arch : Vers. 2 Arch : Sine 2 Arch ::
 2 Cosine Arch : Sine 2 Arch : Vers. Sup. 2 Arch.

For the Triangles CAL, BAF, IBF are similar,
 B 4 and

FIG. and $IB=2CL$, therefore $CA : AL : CL :: BA :$
 1. $AF : BF :: IB : BF : IF$.

Cor. 1. From hence and Prop. I. it is evident, that these are respectively proportional.

$$\begin{array}{llll} \text{Radius} & : & \text{Sine of an Arch} & : \text{Cofine Arch} & :: \\ \text{Secant Arch} & : & \text{Tangent Arch} & : \text{Radius} & :: \\ \text{Cofecant Arch} & : & \text{Radius} & : \text{Cotangent Arch} & :: \\ 2 \text{ Sine Arch} & : & \text{Vers. 2 Arch} & : \text{Sine 2 Arch} & :: \\ 2 \text{ Cofine Arch} & : & \text{Sine 2 Arch} & : \text{Vers. Sup. 2 Arch.} & \end{array}$$

Cor. 2. In any Arch, $\frac{2 \text{ Sine Square}}{\text{Radius}} = \text{versed Sine of the double Arch}$, that is $\frac{2ss}{r} = \text{vers. 2 A}$. See Sch. to Pr. I.

Cor. 3. In any Arch, $\frac{2 \text{ Sine} \times \text{Cofine}}{\text{Radius}} = \text{Sine of the double Arch}$, $\frac{2cs}{r} = \text{S. 2 A}$.

Cor. 4. In any Arch, $\frac{2 \text{ Cofine Square} - \text{Radius Square}}{\text{Radius}} = \text{Cofine of the double Arch}$. For $\text{Cof. 2 A} = r - \text{vers. 2 A} = r - \frac{2ss}{r} = \frac{rr - 2ss}{r} = \frac{2cc - rr}{r}$.

Cor. 5. In any Arch, $\frac{2 \text{ Radius Square}}{\text{Cotangent} - \text{Tangent}} = \text{Tangent of twice the Arch}$. For $\text{Cof. : Sine :: Rad. : Tangent}$, that is by Cor. 4, 3. $\frac{2cc - rr}{r}$ or $\frac{cc - ss}{r} : \frac{2cs}{r} :: r : \frac{2csrr}{cc - ss.r} = \frac{2rr}{\frac{rc}{s} - \frac{rs}{c}} = (\text{by Schol. Pr. I.}) \frac{2rr}{r - t}$.

Cor. 6. In any Arch, $\frac{1}{2} \text{ Cotangent} - \frac{1}{2} \text{ Tangent} = \text{Cotangent of the double Arch}$. This follows from Cor. 5, because Radius Square divided by the Tangent is = Cotangent.

Cor.

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Cor. 7. In any Arch, $\frac{\text{Secant} \times \text{Radius}}{2 \text{ Cofine} - \text{Secant}} = \text{Secant of FIG. 1.}$
the double Arch. For $\frac{\text{Radius Square}}{\text{Cofine}} = \text{Secant}$, that

is by Cor. 4. $\frac{rr}{2cc - rr} r = \frac{\frac{rr}{c}}{2c - \frac{rr}{c}} r = (\text{by Sch. Pr. I.})$

$\frac{rf}{2c - f} = \text{Secant of 2 Arch.}$

Cor. 8. In any Arch, $\frac{\text{Secant} \times \text{Cofecant}}{2 \text{ Radius}} = \text{Cofecant}$
of the double Arch. For by Cor. 1. of this Prop. $\frac{2rs}{f} = \text{S. 2 Arch}$, and (by Sch. Pr. I.) $\frac{\text{Radius Square}}{\text{Sine}} = \text{Cofecant}$, that is $\frac{rrf}{2rs} = \frac{f\sigma}{2r}$, because $\frac{rr}{s} = \sigma$, by Schol. Pr. I.

Cor. 9. In any Arch, $\frac{8 \text{ Sine Square} \times \text{Cofine Square}}{\text{Radius Cube}} =$
versed Sine of the Quadruple Arch. For, by Cor. 3. $\frac{2cs}{r} = \text{S. 2 A}$, and by Cor. 2. $\frac{2 \times 4ccss}{r \times rr} = \text{versed Sine}$
of twice 2 A.

Cor. 10. Hence also, the versed Sines of two Arches are as the Squares of the Sines of their half Arches.

Cor. 11. The Sines of two Arches are as the Rectangles of the Sine and Cofine of their half Arches.

Cor. 12. The Cofecants of two Arches are as the Rectangles of the Secant and Cofecant of their half Arches.

Cor. 13. The versed Sine of the Sup. of an Arch : $\frac{1}{2}$ Radius :: Sine Square : Sine Square of $\frac{1}{2}$ the Arch.

For vers. Sup. 2A : Cof. A :: S. 2A : S.A.

And 2Cof. A : Rad. :: S. 2A : S.A. whence
by multiplying, vers. Sup. 2A : $\frac{1}{2}$ Rad. :: S². 2A : S². A.

Cor.

FIG. 1. *Cor. 14.* What has here been demonstrated of the Relation of Sines and Cosines to one another, holds equally true, in respect of the Chords of Arches, and their Supplements; putting the Diameter instead of Radius.

Cor. 15. Hence also it follows from *Cor. 6.* that the Difference between the Tangent and Cotangent of an Arch = twice the Tangent of the Difference between that Arch and its Complement.

For the Difference between an Arch and its Complement is equal to the Complement of twice that Arch. Hence also

Cor. 16. Let A be any Arch less than 45° , then Tangent of $45 + A^\circ =$ Tangent of $45 - A^\circ + 2$ Tangents of $2A$. For $2A$ is the Difference of the Arches $45 + A$ and $45 - A$.

SCHOLIUM.

Hence also the Sines, Cosines, &c. of the double Arches may be expressed more universally thus, (See *Sch. Pr. I.*)

$$\text{I. } \frac{2cs}{r} = \frac{2ss}{t} = \frac{2cc}{\tau} = \frac{2rs}{f} = \frac{2rc}{\sigma} = \frac{2rrt}{rr+tt=ff} = \frac{2rr}{\tau+t} \\ = \frac{2r^2\tau}{rr+\tau\tau=\sigma\sigma} = \text{Sine of } 2 \text{ Arch.}$$

$$\text{II. } \frac{cc-ss}{r} = \frac{rr-2ss}{r} = \frac{2cc-rr}{r} = \frac{rr-tt}{rr+tt} r = \frac{\tau-t}{\tau+t} r \\ = \frac{\tau\tau-rr}{\tau\tau+rr} r = \frac{2rr-ff}{ff} r = \frac{2c-f}{f} r = \frac{\sigma^2-2r^2}{\sigma^2} r = \frac{\sigma-2s}{\sigma} r \\ = \text{Cosine of } 2 \text{ Arch.}$$

$$\text{III. } \frac{2rrt}{rr-tt} = \frac{2rr}{\tau-t} = \frac{2rcs}{rr-2ss} = \frac{2rcs}{2cc-rr} = \frac{2rr\tau}{\tau\tau-rr} \\ = \text{Tangent of } 2 \text{ Arch.}$$

$$\text{IV. } \frac{rr-tt}{2t} = \frac{\tau\tau-rr}{2\tau} = \frac{\tau-t}{2} = \frac{rr-2ss}{2cs} r = \frac{2cc-rr}{2cs} r \\ = \text{Co-tangent of } 2 \text{ Arch.}$$

$$\text{V. } ff r$$

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II

FIG.

$$\text{V. } \frac{ffr}{2rr-ff} = \frac{fr}{2c-f} = \frac{r^3}{2cc-rr} = \frac{r^3}{rr-2ss} = \frac{rr+tt}{rr-tt} r$$

$$= \frac{r+t}{r-t} r = \frac{rr+rr}{rr-rr} r = \text{Secant of } 2 \text{ Arch.}$$

$$\text{VI. } \frac{f\sigma}{2r} = \frac{r\sigma}{2s} = \frac{r\sigma}{2c} = \frac{r^3}{2cs} = \frac{rr+tt=ff}{2t} = \frac{r+t}{2} =$$

$$\frac{rr+rr=\sigma\sigma}{2r} = \text{Cofecant of } 2 \text{ Arch.}$$

$$\text{VII. } \frac{2ss}{r} = \frac{2rr-2cc}{r} = \frac{2vV}{r} = \frac{2v \times 2r - v}{r} = \frac{2rtt}{rr+tt=ff}$$

$$= \frac{2rt}{r+t} = \frac{2r^3}{rr+rr=\sigma\sigma} = \frac{2rs}{\sigma} = \frac{ff-rr}{ff} \times 2r = \frac{f-c}{f} \times 2r$$

$$= \text{versed sine of } 2 \text{ Arch.}$$

$$\text{VIII. } \frac{2rr-2ss}{r} = \frac{2cc}{r} = \frac{2rr}{rr+tt} r = \frac{2rr}{rr+rr} r = \frac{2rr}{t+t} =$$

$$\frac{2r^3}{ff} = \frac{2cr}{f} = \frac{2\sigma^2-2r^2}{\sigma^2} r = \frac{2\sigma-2s}{\sigma} r = \frac{V-r}{\frac{1}{2}r} = \frac{r-v}{\frac{1}{2}r} =$$

$$\text{vers. Sup. } 2A.$$

The covered Sine may be express'd by any of the Rest, by subtracting the Value of the Sine from Radius. Also other Values of each Quantity may be inserted by Sch. Pr. I.

PROP. III.

If three Arches AF, AN, AP be in arithmetic Progression, then these are respectively proportional. 2.

Radius : S. mean Arch : Cof. mean Arch ::
 2 S. mean Arch : Vers. 2 mean Arch. : S. 2 mean Arch ::
 2 Cof. mean Arch : S. 2 mean Arch. : Vers. Sup. 2 mean Arch ::
 2 S. common Diff. : Diff. Cof. Extrems : Diff. Sin. Extrems ::
 2 Cof. com. Diff. : Sum. Sin. Extrems : Sum Cof. Extrems.

Note, Diff. Sines = S. greater Arch — S. lesser Arch.

Diff. Cosines = Cof. lesser — Cof. greater Arch =

Diff. versed Sines = vers. greater — vers. lesser Arch, universally.

Draw PCM and PF and MF; and AD || FP, and FS || AB, and CE, MS ⊥ AB. Then MS =

FIG. $PG+FH$, and $SF=CG+CH$. And Arch $OM=$
 2. $PN=NF$. Therefore $MF \parallel NC$, and $MF=2CR$.
 Whence the Triangles CAQ , DAK , BDK , PFI ,
 FMS are similar; and $CA : AQ : CQ :: DA :$
 $AK : DK :: BD : DK : BK :: PF : FI : PI ::$
 $FM : MS : FS$. Q. E. D.

Cor. 1. Since the mean $= \frac{1}{2}$ Sum of the Extreams,
 therefore in any two Arches,

Radius	: S. $\frac{1}{2}$ Sum	: Cof. $\frac{1}{2}$ Sum	::
2 S. $\frac{1}{2}$ Sum	: Verf. Sum	: S. Sum	::
2 Cof. $\frac{1}{2}$ Sum	: S. Sum	: Verf. Sup. Sum	::
2 S. $\frac{1}{2}$ Difference	: Diff. Cofines	: Diff. Sines	::
2 Cof. $\frac{1}{2}$ Diff.	: Sum Sines	: Sum Cofines.	

Cor. 2. Let A , E be two Arches, A the greater, E
 the lesser, then

$$\begin{aligned} \frac{1}{2} \text{ Rad.} : S. A & : \text{Cof. } A :: \\ S. E : \text{Cof. } \overline{A-E} - \text{Cof. } \overline{A+E} : S. \overline{A+E} - S. \overline{A-E} :: \\ \text{Cof. } E : S. \overline{A+E} + S. \overline{A-E} : \text{Cof. } \overline{A-E} + \text{Cof. } \overline{A+E}. \end{aligned}$$

This will appear by Cor. 1. putting $A =$ half the
 Sum of 2 Arches, and $E =$ half their Difference. For
 then the Arches will be $\overline{A+E}$, and $\overline{A-E}$.

Cor. 3. As $\frac{1}{2}$ Rad. : S. $\frac{1}{2}$ Sum of two Arches :: S.
 $\frac{1}{2}$ Difference : Difference of their Cofines, or of their
 versed Sines.

Cor. 4. Rectangle of Radius and the Difference of
 the versed Sines (or Cofines) of two Arches $=$ twice
 the Rectangle of the Sine of half the Sum, and Sine
 of half the Difference of these Arches.

Cor. 5. If three Arches AF , AN , AP are in
 arithmetic Progression, the Rectangle of the Sine
 of the mean Arch, and versed Sine of the common
 Difference $=$ Rectangle of Radius, and the Difference
 between the Sine of the mean Arch, and half the
 Sum of the Sines of the extream Arches.

For $CA \times \overline{PG+FH} = 2CR \times AQ = 2AQ \times \overline{CN-NR}$,
 whence $2AQ \times NR = 2AQ \times CA - CA \times \overline{PG+FH}$.
 Hence,

Cor.

Cor. 6. The second Difference of the Sines FH, AQ, FIG. PG, $\times$ Radius = twice the Sine of the mean Arch $\times$ 2. versed Sine of the common Difference. For the second Difference = $2 A Q - FH - PG$.

Cor. 7. As Radius :

To twice the Cosine of any Arch ::

So the Sine of n times that Arch :

To Sum of the Sines of $n-1$ and $n+1$ times that Arch ::

And so the Cosine of n times that Arch :

To the Sum of the Cosines of $n-1$ and $n+1$ times that Arch.

For if A be the Arch, then $n-1.A$, $n.A$, $n+1.A$ are in arithmetic Progression, and A the common Difference.

Cor. 8. Let A be any Arch less than 30° , then Sine of $A \times \sqrt{3} +$ Sine of $30-A =$ Sine of $30+A$.

For the Arches $30-A$, 30° , and $30+A$ are in arithmetic Progression, and Cos. mean $30 = \frac{1}{2}\sqrt{3} \times$ Radius. And by this Prop. Rad. : Cos. $30^\circ :: 2$ Sine of $A : S. 30+A - S. 30-A$.

Cor. 9. If A be any Arch less than 30 , then $S. 30+A + S. 30-A = S. 90-A$.

For the S. 30 the mean $= \frac{1}{2}$ Rad. and by this Prop. Rad. : $\frac{1}{2}$ Rad. :: 2 Cos. A : $S. 30+A + S. 30-A =$ Cos. A.

Cor. 10. If A be any Arch less than 30° , then $S. 60+A \times \sqrt{3} - S. 30+A = S. 90-A$.

For if the Arches be A, $60+A$, $120+A$, and Radius $\times \frac{1}{2}\sqrt{3} = S. 60$ the common Difference. Then Rad. : Rad. $\times \sqrt{3} :: S. 60+A : \text{Cos. } A - \text{Cos. } 120+A = \text{Cos. } A + \text{Cos. } 60-A$, that is $1 : \sqrt{3} :: S. 60+A : S. 90-A + S. 30+A$.

Cor. 11. If A be any Arch less than 45° ; then $S. A \times \sqrt{2} + S. 45-A = S. 45+A$. For in the Arches $45-A$, 45 , $45+A$; $\frac{1}{2}\sqrt{2} \times$ Rad. = Cos. the mean, 45 .

Cor. 12. Let A be any Arch less than 60 ; then $S. A +$

FIG. S. $A + S. 60 - A = S. 60 + A$. For in the Arches $60 - A$, 60 , $60 + A$, the Cos. mean, $60^\circ = \frac{1}{2}$ Rad.

Cor. 13. What is demonstrated in this Prop. and Cor. 2, 3, 7, of Sines and Cosines, holds of Cords of Arches, and their Supplements, putting Diameter for Radius.

P R O P. IV.

In any two Arches, these are respectively proportional.

$\frac{1}{2}$ Radius. : S. $\frac{1}{2}$ Sum : Cos. $\frac{1}{2}$ Sum : S. $\frac{1}{2}$ Diff. : Cos. $\frac{1}{2}$ Diff. ::
 S. $\frac{1}{2}$ Sum : Vers. Sum : S. Sum : Diff. Cos. : Sum Sines ::
 Cos. $\frac{1}{2}$ Sum : S. Sum : V. Sup. Sum : Diff. Sines : Sum Cos. ::
 S. $\frac{1}{2}$ Diff. : Diff. Cos. : Diff. Sines : Vers. Diff. : S. Diff. ::
 Cos. $\frac{1}{2}$ Diff. : Sum Sines : Sum Cos. : S. Diff. : V. Sup. Diff.

Note, *Diff. Sines and Cosines, is the same as in Pr. III.*

The 3 first perpendicular Rows are evident from the last Prop. and the first, fourth, and fifth Terms in the 2 last perpendicular Rows follow from Prop. II. and the remaining Terms are fill'd up by the Proportions in this very Prop. thus, $\frac{1}{2}$ Rad. : S. $\frac{1}{2}$ Diff. :: S. $\frac{1}{2}$ Sum : Diff. Cosines, which therefore will stand in the second Place of the fourth perpendicular Column, as well as in the fourth Place of the second Column, &c.

Hence several Corollaries follow of their own accord ; as

Cor. 1. As vers. Sup. Sum of 2 Arches : Diff. Sines :: Diff. Sines : versed Sine of their Difference.

Cor. 2. Sine of the Sum of 2 Arches : Sum of their Sines :: Difference of their Sines : Sine of the Difference of the Arches.

Cor. 3. If three Arches are in arithmetical Progression,

As the Sine of one Extream :

Sum of the Sines of the mean Arch, and common Difference ::

So their Difference (S. mean — S. com. Diff.) :

To the Sine of the other Extream.

For let m = mean Arch, d = com. Difference, and then

then by Cor. 2. $S. \overline{m+d} : S. m + S. d :: S. m - S. d : S. \overline{m-d}$. FIG.

Cor. 4. Let A be any Arch; m, n any Numbers, then

As Sine of $\overline{m-n} . A :$

$S. mA + S. nA ::$

$S. mA - S. nA :$

Sine of $\overline{m+n} . A$. By this Prop. hence

Cor. 5. As the Sine of an Arch :

Sum of the Sines of the double and single Arch ::

So their Difference (S. double Arch — S. single Arch) :

To Sine of the tripple Arch.

Cor. 6. As the Sine of an Arch :

Sum of the Sines of the triple and double ::

So their Difference (S. triple — S. double) :

To Sine of the Quintriple Arch, &c.

Cor. 7. Let $r =$ Radius, $s =$ Sine of the Arch $2A$,

then the Sine of $\overline{45+A} = \sqrt{\frac{rr+rs}{2}}$, and the Sine of

$$\overline{45-A} = \sqrt{\frac{rr-rs}{2}}.$$

For let $a = S. \overline{45+A}$. $e = S. \overline{45-A}$. Then by this Prop. as (S. Sum) $r : (\text{Sum Sines}) a+e :: (\text{Diff. Sines}) a-e : s$ the Sine of the Difference. Therefore

$$rs = aa - ee$$

but $rr = aa - ee$ by Cor. 1. Prop. I.

therefore $rr + rs = 2aa$

and $rr - rs = 2ee$.

Cor. 8. What has been demonstrated in this Prop. and first 6 Corollaries concerning the Relation of Sines, and Cosines, holds equally true with respect to their Cords; putting the Cord of the Supplement instead of Cosine, and the Diameter instead of Radius.

SCHO-

FIG.

SCHOLIUM.

Let there be two Arches A, E, and let
 $C = \text{Cof. } \overline{A+E} . c = \text{Cof. } \overline{A-E}$

$V = \text{Verf. } \overline{A+E} . v = \text{Verf. } \overline{A-E}$

$Z = \text{Sine } \frac{A+E}{2} . z = \text{S. } \frac{A-E}{2}$

$Y = \text{Cof. } \frac{A+E}{2} . y = \text{Cof. } \frac{A-E}{2}$

$D = \text{Cof. } \overline{2A+2E} . d = \text{Cof. } \overline{2A-2E}$, Then

$$\begin{aligned} S. A \times S. E &= \overline{Z+z} \times \overline{Z-z} = Z^2 - z^2 = \overline{y+Y} \times \overline{y-Y} \\ &= y^2 - Y^2 = \frac{V-v}{2} r = \frac{c-C}{2} r. \end{aligned}$$

$$\text{Cof. } A \times \text{Cof. } E = \frac{c+C}{2} r.$$

$$\text{Tan. } A \times \text{Tan. } E = \frac{c-C}{c+C} r r.$$

$$S. A \times S. E \times \text{Cof. } A \times \text{Cof. } E = \frac{cc-CC}{4} r r = \frac{d-D}{8} r^3.$$

$$\text{Cof. } A \times \text{Cof. } E \times \text{Tan. } A \times \text{Tan. } E = \frac{c-C}{2} r^3.$$

PROP. V.

3. *In any two Arches (AD, DE), the Sum of the Rectangles of the Sine of one into the Cosine of the other, is = Rectangle of the Radius, and the Sine of the Sum of these Arches.*

DEMONSTRATION.

Draw $NF \parallel$ and $NR \perp CA$: then because the Angles ENC , FNR are right, therefore $ENF = CNR$; and therefore the Triangles ENF , CNR and CDG are similar; wherefore $CD : DG :: CN : NR = \frac{DG \times CN}{CD} = FI$. And $CD : CG :: EN :$

$EF =$

$$EF = \frac{CG \times EN}{CD}, \text{ whence } EI = EF + FI =$$

$$\frac{CG \times EN + DG \times CN}{CD}$$

Q. E. D.

Cor. 1. In any two Arches (A D, D E), the Rectangle of their Cosines — the Rectangle of their Sines = the Rectangle of Radius and the Cosine of their Sum.

For by similar Triangles, $CD : CG :: CN : CR = \frac{CG \times CN}{CD}$. And $CD : DG :: (CN : NR ::) EN :$

$$NF = \frac{DG \times EN}{CD} = IR. \text{ And } CI = CR - IR = \frac{CG \times CN - DG \times EN}{CD}.$$

Cor. 2. The Square of the Sine of the Sum of two Arches is = the Sum of the Squares of their Sines + twice their Rectangle multiply'd by the Cosine of the Sum and divided by Radius.

For let the Arch $A + E = S$, their Sines a, e, s ; and c the Cosine of S . Then by this Prop. $s = \frac{\sqrt{rr - ee} + e\sqrt{rr - aa}}{r}$ and $ss = aa + ee - \frac{2aee}{rr} + \frac{2ae}{rr}$

$\times \sqrt{rr - aa} \times \sqrt{rr - ee}$. But by Cor. 1. $\frac{\sqrt{rr - aa} \times \sqrt{rr - ee}}{r}$

$$- \frac{ae}{r} = c. \text{ Therefore } ss = aa + ee + \frac{2c}{r} ae.$$

Cor. 3. Hence if $A + E = 90^\circ$, then $aa + ee = rr$.

If $A + E = 60$, then $aa + ee + ae = \frac{3}{4}rr$.

If $A + E = 45$, then $aa + ee + ae\sqrt{2} = \frac{1}{2}rr$.

If $A + E = 30$, then $aa + ee + ae\sqrt{3} = \frac{1}{4}rr$.

SCHOLIUM.

This Prop. and Cor. 1. holds universally for all Arches, putting negative Cosines for Arches greater than a Quadrant.

C

PROP.

FIG.

3.

PROP. VI.

In two Arches (AD, AE), the Rectangle of the Sine of the greater and Cosine of the lesser — the Rectangle of the Sine of the lesser and Cosine of the greater is = Rectangle of Radius and the Sine of the Difference of the Arches.

DEMONSTRATION.

The Triangles CDG, CBI, and EBN are similar; therefore $CG : DG :: CI : BI = \frac{CI \times DG}{CG}$

And $CD : CG :: EB \text{ or } EI - \frac{CI \times DG}{CG} : EN = \frac{CG \times EI - CI \times DG}{CD}$

Cor. In any two Arches (AD, AE), the Rectangle of their Sines + the Rectangle of their Cosines is = Rectangle of the Radius and the Cosine of the Difference of the Arches.

For $BI = \frac{CI \times DG}{CG}$, and by similar Triangles $CB = \frac{CI \times CD}{CG}$. And $CD : DG :: BE \text{ or } EI - \frac{DG \times CI}{CG} : BN = \frac{DG \times EI}{CD} - \frac{DG \times CI}{CD \times CG} = \frac{DG \times EI}{CD} - \frac{CD^2 \times CI}{CD \times CG} + \frac{CG^2 \times CI}{CD \times CG} = \frac{DG \times EI}{CD} - CB + \frac{CG \times CI}{CD}$
therefore $\frac{DG \times EI + CG \times CI}{CD} = CB + BN = CN$.

SCHOLIUM I.

This Prop. and its Cor. holds universally for all Arches, putting negative Cosines for Arches greater than a Quadrant.

SCHO

From the foregoing Propositions it follows, that if
Radius = 1. And

S, C = Sine, and Cosine of half the Sum of two
Arches

s, c = Sine, and Cosine of half their Difference,
then

$S c + C s =$ Sine of the greater.

$S c - C s =$ Sine of the lesser.

$C c - S s =$ Cosine of the greater.

$C c + S s =$ Cosine of the lesser.

Also,

If Z = Secant of the greater Arch, z = Secant of
the lesser.

Y = Cosecant of the greater, y = Cosecant of
the lesser.

Then $\frac{Y y Z z}{Y y - Z z} =$ Secant of the Sum.

$\frac{Y y Z z}{Z y + Y z} =$ Cosecant of the Sum.

$\frac{Y y Z z}{Y y + Z z} =$ Secant of the Difference.

$\frac{Y y Z z}{Z y - Y z} =$ Cosecant of the Difference.

P R O P. VII.

If three Arches (A F, A N, A P) be in arithmetic
Progression, 4.

The Sum of the Sines of the Extream Arches :

To their Difference (S. greater — S. lesser) ::

As Tang. mean Arch :

To Tang. common Difference of the Arches.

For drawing the Lines as in the Figure, then by
similar Triangles, 2 O B or P G + F H : 2 O D or P I
:: O B : O D :: O L : O F :: N M : N K.

Cor. 1. In any two Arches (A F, A P ;)

As Sum of their Sines :

To their Difference (S. greater — S. lesser) ::

C 2

Tang.

FIG.

4.

*Tang. half their Sum :**Tang. half their Difference.*For the mean Arch $= \frac{1}{2}$ Sum of the Extrems.

Cor. 2. In any two Arches (A F, A P,)

*The Sum of their Cosines :**Their Diff. (Cos. lesser — Cos. greater) ::**As Cotang. half their Sum :**Tang. half their Difference.*For $CH + CG$ or $2 CB : CH - CG$ or $2 DF ::$
 $CB : BH :: RO : OF :: SN : NK.$

Cor. 3. In any two Arches (A F, A P,) as

 $S. \text{ greater} : S. \text{ lesser} :: T. \frac{1}{2} \text{ Sum} + T. \frac{1}{2} \text{ Diff.} : T. \frac{1}{2} \text{ Sum}$
 $- T. \frac{1}{2} \text{ Diff.}$ For by the Demonstration of this Prop. $PG + FH :$
 $PG - FH :: NM : NK.$ And by Composition and
Division $PG : FH :: QM : KM.$

Cor. 4. In any two Arches (A N, N P,) as

*The Sum of their Tangents :**Their Difference, (Tan. greater — Tan. lesser) ::**Sine of their Sum :**Sine of their Difference.*Let A N be the greater, and make $NF = NP,$
then by Similar Triangles, $QM : KM :: (PL : FL ::)$
 $PG : FH.$

Cor. 5. In any two Arches (A N, N P,) as

 $Tan. \text{ greater} : Tan. \text{ lesser} :: S. \text{ Sum} + S. \text{ Difference} :$
 $S. \text{ Sum} - S. \text{ Difference.}$ For let A N be the greater, and make $NF = NP,$
then A P is the Sum, and, A F is the Difference
of the Arches ; and by Cor. 4. we have $QM : KM$
 $:: PG : FH.$ And by Composition and Division,
 $\frac{QM + KM}{2}$ or $NM : \frac{QM - KM}{2}$ or $NQ :: PG +$
 $FH : PG - FH.$ Cor. 6. Let A be any Arch less than 45° , then $S. 45 + A + Cos. 45 + A :$ $S. 45 + A - Cos. 45 + A ::$

Radius

Radius :

Tan. A.

This appears by this Prop. putting $AN=45^\circ$, and $NF=NP=A$.

SCHOLIUM.

The Propositions here delivered hold universally, observing to take negative Cosines and Tangents for Arches greater than a Quadrant.

PROP. VIII.

In any two Arches, AE, ED;

Radius Square — Rectangle of their Tangents :

Radius Square ::

Sum of their Tangents :

Tangent of the Sum of the Arches.

5.

For by similar Triangles CB or CA—BA : CK or CF—KF :: CF : CA. Whence $CA^2 - BAC = CF^2 - KFC$, and $KFC = CF^2 - CA^2 + BAC = FA^2 + BAC$. Again CB : BK :: CF : FA, and BK : DG :: KF : FD or FA, therefore *ex equo* CB or CA—AB : DG :: CFK or $FA^2 + BAC : AF^2$; therefore $AC - AB \times AF^2 = DG \times FA^2 + DG \times BAC$. And $AB + DG \times AF^2 = AC \times AF^2 - AB \times DG$. Q. E. D.

Cor. 1. Let T, t be the Tangents, X, x, the Cotangents of two Arches, then $\frac{rr - Tt}{T+t} = \frac{Xx - rr}{X+x} = \text{Cotangent of the Sum of the Arches}$.

For let $r = \text{Radius}$, $\tau = \text{Cotangent of the Sum}$.

Then $AC = \frac{rr}{\tau}$, and $AB + DG \times rr = \frac{rr}{\tau} \times rr - AB \times DG$, whence $\tau = \frac{rr - AB \times DG}{AB + DG} = \frac{\text{rect. Cot.} - rr}{\text{Sum Cot.}}$

FIG. Cor. 2. If one of the Arches be 45° , $t = \text{Tang.}$ other Arch, then $\frac{r+t}{r-t} \times r = \text{Tang. Sum of the Arches.}$

S C H O L I U M.

What belongs in general to the Addition of Tangents and Cotangents, is delivered in this Prop. and its Cors. All Cases where the Arches are greater than a Quadrant will easily appear by putting negative Tangents and Cotangents for these Arches.

P R O P. IX.

5. *In any two Arches AD, AE, as*
Radius Square + Rectangle of their Tangents :
Radius Square ::
Difference (or Tangent greater — Tangent lesser) :
Tangent of the Difference of the Arches.

For you will find as in the last Prop. $CB : DG :: CFK \text{ or } FA^2 + BAC : FA^2.$ *Q. E. D.*

Cor. 1. Let T, t be the Tangents, X, x the Cotangents of two Arches; then $\frac{rr + Tt}{T - t} = \frac{rr + Xx}{x - X} = \text{Cotangent of the Difference of the Arches.}$

This appears by putting $\frac{rr}{t}$ for the Tangent of the Difference.

Cor. 2. If one of the Arches be 45° , $t = \text{Tangent}$ of the other, than $\frac{r-t}{r+t} r = \text{Tangent of their Difference} = \text{Tan. } 45^\circ - \text{Tan. of the other Arch.}$

S C H O L I U M.

For Arches greater than a Quadrant, put negative Tangents and Cotangents.

P R O P.

PROP. X.

The Secant of an Arch is equal to the Sum of the Tangent of it, and the Tangent of half its Complement.

For let A = Arch, T its Tangent, f its Secant, $a = \frac{1}{2}$ Complement, t its Tangent. Then by Schol.

Pr. II. $f = \frac{rr+tt}{2t}$, and $T = \frac{rr-tt}{2t}$, and $f - T = t$,

whence Secant of A = Tangent of $A + \text{Tang. } \frac{1}{2} \text{ Complement of } A$.

Cor. 1. The Secant of an Arch = Cotangent of half the Complement — the Tangent of the Arch.

For $S + T = \frac{rr}{t} = \text{Cot. } a = \text{Cotang. } \frac{1}{2} \text{ Compl. of } A$.

Cor. 2. Half the Sum of the Tangent and Cotangent of an Arch is = Secant of the Difference between that Arch and its Complement.

For by Schol. Pr. II. $\frac{1}{2} \text{ Tangent} + \frac{1}{2} \text{ Cotangent} = \text{Cosecant of the double Arch, and the Complement of the double Arch is the same as the Difference between the Arch and its Complement.}$

Cor. 3. As Radius + Secant : Radius :: Tangent : Tangent of half the Arch. For $r + \frac{r+t}{r-t}r$ or $\frac{2rr}{r-t}$; r

:: $\frac{2rr}{r-t}$; t . See Sch. Pr. II.

Cor. 3. Secant of an Arch A = Tangent of $45 + \frac{1}{2}A$ — Tangent of A . This follows from Cor. 1.

SCHOLIUM.

From what has been before laid down, it will not be difficult to find the Sines of as many Arches as we will, express'd in furd Numbers. As in these following, where the Prop. in the Margin shows whence and how they are derived, either immediately, or by help of what Step.

FIG. Pr. IV, 7. Step 8.	1	S. $15^\circ = \frac{1}{2}r\sqrt{2-\sqrt{3}}$.
Pr. IV, 7. and II. 3.	2	S. $18 = r \times \frac{\sqrt{5-1}}{4}$.
Pr. IV, 7; Step 6.	3	S. $22 \frac{1}{2} = \frac{1}{2}r\sqrt{2-\sqrt{2}}$.
Pr. III, Step 12.	4	S. $30 = \frac{1}{2}r$.
Pr. IV, 7. Step 2.	5	S. $36 = \frac{1}{4}r\sqrt{10-2\sqrt{5}}$.
Pr. IV, 7.	6	S. $45 = \frac{1}{2}r\sqrt{2}$.
Pr. IV, 7. Step 2.	7	S. $54 = r \times \frac{\sqrt{5+1}}{4}$.
Pr. IV, 7. Step 4.	8	S. $60 = \frac{1}{2}r\sqrt{3}$.
Pr. IV, 7. Step 6.	9	S. $67 \frac{1}{2} = \frac{1}{2}r\sqrt{2+\sqrt{2}}$.
Pr. I, 1. Step 2.	10	S. $72 = \frac{1}{4}r\sqrt{10+2\sqrt{5}}$.
Pr. IV, 7. Step 8.	11	S. $75 = \frac{1}{2}r\sqrt{2+\sqrt{3}}$.
Def. 2.	12	S. $90 = r$.

And thus you may find as many Sines as you please by Prop. III. Cor. 6, 7, 8, 9. And Prop. IV. Cor. 7. and Prop. V. Cor. 4, 5. by the help of these already found; but then they will be still more and more compounded with surd Quantities, except you chuse to extract them. The Sine of 18 in Step the second is not so easily derived as the Rest, it is had thus.

Put $s = S.18$. Then by Pr. IV. 7. $S.36 = \sqrt{\frac{r-rs}{2}}$,

and by Prop. II. 3, the same Sine of $36 = \frac{2s}{r}\sqrt{rr-ss}$;

which equated and divided by $\sqrt{r-s}$, and then reduced, you have a Cubic Equation $8s^3 + 8rss = r^3$

And the Root is $s = \frac{\sqrt{5-1}}{4} \times r$.

From what has been said and laid down, it will not be difficult to find the Sines of as many Arcs as we will, expressed in surd Numbers. As in the following, where the Root in the Margin shows whence and how they are derived, either immediately, or by help of what Step.

S E C T. II.

The Calculation of natural Sines, Tangents, and Secants of Arches.

P R O P. XI.

The Radius and Sine AB of an Arch DB being given; to find the Arch. FIG. 6.

Take the Arch Bb infinitely small, and draw $Bs \parallel CD$, and $bsu \parallel BA$; and let CB or $CD = r$, $AB = y$, $DB = z$, $CA = \sqrt{rr - yy}$, $Bb = \dot{z}$, $sb = \dot{y}$. The Triangles CAB , and Bsb are similar: Therefore $CA (\sqrt{rr - yy}) : CB (r) :: sb (\dot{y}) : Bb$ or $\dot{z} = \frac{r\dot{y}}{\sqrt{rr - yy}}$. And the Fluent is $z = y + \frac{y^3}{3 \cdot 2r^2} + \frac{3 \cdot y^5}{5 \cdot 2 \cdot 4r^4} + \frac{3 \cdot 5y^7}{7 \cdot 2 \cdot 4 \cdot 6r^6} + \frac{3 \cdot 5 \cdot 7y^9}{9 \cdot 2 \cdot 4 \cdot 6 \cdot 8r^8} \&c.$ or Arch $DB = y + \frac{yy}{2 \cdot 3rr} A + \frac{3 \cdot 3y^2}{4 \cdot 5rr} B + \frac{5 \cdot 5y^2}{6 \cdot 7r^2} C + \frac{7 \cdot 7y^2}{8 \cdot 9r^2} D \&c.$ putting $A, B, C, \&c.$ for the first, second, third Terms, $\&c.$

Cor. 1. If $d =$ Diameter, $c =$ Cord of an Arch, then the Arch $= c + \frac{cc}{2 \cdot 3dd} A + \frac{3c^2}{4 \cdot 5d^2} B + \frac{5c^2}{6 \cdot 7d^2} C \&c.$

Cor. 2. Let $Q =$ a Quadrant, then the Arch whose Cosine is y is $= Q - y - \frac{y^3}{2 \cdot 3r^2} - \frac{3y^5}{2 \cdot 4 \cdot 5r^4} - \frac{3 \cdot 5y^7}{2 \cdot 4 \cdot 6 \cdot 7r^6} \&c.$

Or the Arch whose Cosine is y is $= \frac{r - y}{1} + \frac{r^3 - y^3}{2 \cdot 3r^2} + 3 \times \frac{r^5 - y^5}{2 \cdot 4 \cdot 5r^4} + 3 \times 5 \times \frac{r^7 - y^7}{2 \cdot 4 \cdot 6 \cdot 7r^6} \&c.$

P R O P.

FIG.

P R O P. XII.

6.

The Arch BD being given; to find the Sine BA.

Let $CD=r$, $AB=y$, $DB=a$. Then by Prop. XI.
 $a=y+\frac{y^3}{6r^2}+\frac{3y^5}{40r^4}+\frac{5y^7}{112r^6}+\mathcal{E}c$. By the Method
 of Reversion of Series, suppose $y=ba+ca^3+da^5+ea^7+\mathcal{E}c$. Then

$$\left. \begin{aligned} y &= ba + ca^3 + da^5 + ea^7 + \mathcal{E}c. \\ + \frac{y^3}{6rr} &= \frac{b^3a^3}{6rr} + \frac{cb^2}{2rr}a^5 + \frac{b^2d+ccb}{2rr}a^7 + \mathcal{E}c. \\ + \frac{3y^5}{40r^4} &= \frac{3b^5}{40r^4}a^5 + \frac{3cb^4}{8r^4}a^7 + \mathcal{E}c. \\ + \frac{5y^7}{112r^6} &= \frac{5b^7}{112r^6}a^7 + \mathcal{E}c. \end{aligned} \right\} = a.$$

then equating the homologous Powers of a , $ba=a$,
 and $b=1$. Also $c+\frac{1}{6rr}=0$, and $c=-\frac{1}{6rr}=\frac{-1}{2.3r^2}$.

Also $d+\frac{c}{2rr}+\frac{3}{40r^4}=0$. Whence $d=\frac{1}{120r^4}=\frac{1}{2.3.4.5r^4}$. Likewise $e=\frac{-1}{5040r^6}=\frac{-1}{2.3.4.5.6.7r^6}$, $\mathcal{E}c$.

Whence y or $AB=a-\frac{a^3}{2.3rr}+\frac{a^5}{2.3.4.5r^4}-\frac{a^7}{2.3.4.5.6.7r^6}+\mathcal{E}c$.

Or Sine $AB=a-\frac{aa}{2.3r^2}A-\frac{aa}{4.5r^2}B-\frac{aa}{6.7r^2}C-\frac{aa}{8.9r^2}D$, $\mathcal{E}c$. where $A, B, C, \mathcal{E}c$. are the foregoing
 Terms with their Signs.

Cor. 1. If a be any Arch, its Cosine $CA=r-\frac{aa}{1.2r^2}A-\frac{aa}{3.4r^2}B-\frac{aa}{5.6r^2}C-\frac{aa}{7.8r^2}D-\mathcal{E}c$.

For

$$\text{For } CA = \sqrt{rr - yy} = \sqrt{rr - a^2 + \frac{a^4}{3r^2} - \frac{2a^6}{45r^4} + \frac{a^8}{315r^6}}$$

$$- \mathfrak{S}c. = r - \frac{aa}{1.2r} + \frac{a^4}{1.2.3.4r^3} - \frac{a^6}{1.2.3.4.5.6r^5} + \frac{a^8}{1.2....8r^7} - \mathfrak{S}c.$$

Cor. 2. If d be the Diameter, a any Arch; then the Cord $= a - \frac{aa}{2.3d^2} A - \frac{a^3}{4.5d^2} B - \frac{a^5}{6.7d^2} C - \frac{a^7}{8.9d^2} D, \mathfrak{S}c.$

P R O P. XIII.

The Sine and Cosine of an Arch A being given, and if there be given another Arch z ; to find the Sine and Cosine of the Sum of the Arches $A+z$, and also of the Difference $A-z$.

Let s, c be the Sine and Cosine of A . And by Prop. XII. and Cor. 1. the Sine of the Arch z is $z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4}, \mathfrak{S}c. = B$. And its Cosine $r - \frac{z^2}{2r} + \frac{z^4}{24r^3}$

$\mathfrak{S}c. = D$. Then by Prop. V. and Cor. 1. and Prop. VI. and Cor. $\frac{sD+cB}{r} = S.A+z$. And $\frac{cD-sB}{r} =$

Cof. $A+z$. And $\frac{sD-cB}{r} = S.A-z$. And $\frac{sB+cD}{r} =$

Cof. $A-z$, that is,

$$S.A+z = s + \frac{cz}{r} - \frac{sz^2}{1.2r^2} - \frac{cz^3}{2.3r^3} + \frac{sz^4}{1.2.3.4r^4} + \frac{cz^5}{1.2.3.4.5r^5} - \mathfrak{S}c.$$

$$\text{Cof. } A+z = c - \frac{sz}{r} - \frac{cz^2}{2r^2} + \frac{sz^3}{2.3r^3} + \frac{cz^4}{2.3.4r^4} - \frac{sz^5}{2.3.4.5r^5} - \mathfrak{S}c.$$

$$S.A-z$$

FIG.

$$\text{S. } \overline{A-z} = s - \frac{cz}{r} - \frac{sz^2}{2r^2} + \frac{cz^3}{2.3r^3} + \frac{sz^4}{2.3.4r^4} - \frac{cz^5}{2.3.4.5r^5} - \mathcal{E}c.$$

$$\text{Cos. } \overline{A-z} = c + \frac{sz}{r} - \frac{cz^2}{2r^2} - \frac{sz^3}{2.3r^3} + \frac{cz^4}{2.3.4r^4} + \frac{sz^5}{2.3.4.5r^5} - \mathcal{E}c.$$

Cor. 1. If there be given s, c , the Sine and Cosine of an Arch A ; and $s+x$ be the Sine of another Arch $\overline{A+z}$; then the Difference of the Sines $x = \frac{cz}{r} - \frac{sz^2}{2r^2}$

$$- \frac{cz^3}{2.3r^3} + \frac{sz^4}{2.3.4r^4}, \mathcal{E}c. \text{ And the Difference of the Arches } z = \frac{r}{c} \times x + \frac{sx^2}{2c^2} + \frac{rr+2ss}{6c^4}x^3 + \frac{3r^2+2ss}{8c^6}sx^4 \mathcal{E}c.$$

For by this Prop. the Sine of $\overline{A+z}$, or $s+x = s + \frac{cz}{r} - \frac{sz^2}{2rr} \mathcal{E}c.$ and $x = \frac{cz}{r} - \frac{sz^2}{2rr} \mathcal{E}c.$ and by Reversion of Series z is found as above.

Cor. 2. The second Difference of the Sines of $\overline{A-z}$, A , $\overline{A+z}$, is $= 2s \times : \frac{z^2}{1.2r^2} - \frac{z^4}{1.2.3.4r^4} + \frac{z^6}{1.2.3.4.5.6r^6} - \mathcal{E}c.$ (for in any three Things, the second Difference is equal to the Difference between twice the mean, and the Sum of the Extrems.)

P R O P. XIV.

6. *The versed Sine of an Arch DB being given; to find the Arch.*

Let $CD=r$. $DA=v$. $Au=\dot{v}$. $DB=z$. Then $AB=\sqrt{2rv-vv}$. The Triangles CAB and Bsb are similar, therefore $\sqrt{2rv-vv} : r :: \dot{v} : Bb$ or $\dot{z} = \frac{r\dot{v}}{\sqrt{2rv-vv}}$. And the Fluent is $z = \sqrt{2rv} \times : 1 +$

$\frac{v}{3 \cdot 2^2 r} + \frac{2v^2}{2^3 \cdot 4 \cdot 5 r^2} + \frac{3 \cdot 5 v^3}{2^4 \cdot 4 \cdot 6 \cdot 7 r^3}$, &c. that is Arch DB FIG.

$$= \sqrt{2rv} + \frac{v}{3 \cdot 4 r} A + \frac{3^2 v}{5 \cdot 8 r} B + \frac{5^2 v}{7 \cdot 12 r} C + \frac{7^2 v}{9 \cdot 16 r} D, \text{ \&c.}$$

$$\text{Or rather } DB = \sqrt{dv} + \frac{v}{2 \cdot 3 d} A + \frac{3^2 v}{4 \cdot 5 d} B + \frac{5^2 v}{6 \cdot 7 d} C \text{ \&c.}$$

putting $d = \text{Diameter}$. Where A, B, C are the foregoing Terms.

Cor. 1. In any Arch a , the versed Sine is $= \frac{aa}{2r} -$

$$\frac{aa}{3 \cdot 4 r^2} A - \frac{aa}{5 \cdot 6 r^2} B - \frac{aa}{7 \cdot 8 r^2} C - \text{ \&c. where } A, B, C$$

are the foregoing Terms with their Signs.

For Radius — Cosine is equal to the versed Sine,

$$\text{and by Cor. 1. Pr. XII. the Cosine is } r - \frac{a^2}{1 \cdot 2 r} +$$

$$\frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4 r^3}, \text{ \&c.}$$

Cor. 2. The covered Sine of the Arch a , is $= r -$

$$a + \frac{a^3}{2 \cdot 3 r^2} - \frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5 r^4} + \frac{a^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 r^6}, \text{ \&c. for it is}$$

$= \text{Radius} - \text{Sine}$.

Cor. 3. Versed Sine of the Supplement is $= 2r$

$$- \frac{a^2}{1 \cdot 2 r^2} A - \frac{aa}{3 \cdot 4 r^2} B - \frac{aa}{5 \cdot 6 r^2} C - \frac{aa}{7 \cdot 8 r^2} D, \text{ \&c. for}$$

it is $= \text{Radius} + \text{Cosine}$.

PROP. XV.

The Tangent DT of an Arch DB being given; to find the Arch.

6.

For draw Ct infinitely near CT , and $Tn \perp Ct$, and let $CD=r$, $DT=t$, $Tt=\dot{t}$, $DB=z$, $CT=\sqrt{rr+tt}$.

The Triangles CDT and Ttn are similar, as also CBb , and CTn , whence $CT : r :: \dot{t} :: zT = \frac{r\dot{t}}{CT}$, and

FIG. 6. and $CT : nT$ or $\frac{rt}{CT} :: r : Bb = \frac{rrt}{CT^2}$, that is $z =$

$$\frac{rrt}{rr+tt}$$

And the Fluent is $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \dots$, &c.
that is, Arch DB = $t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \dots$, &c.

Cor. 1. In any Arch a the Tangent is $= a + \frac{a^3}{3rr} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} + \frac{1382a^{11}}{155925r^{10}} + \frac{21844a^{13}}{6081075r^{12}} + \frac{929569a^{15}}{638512875r^{14}} + \dots$, &c.

This will appear by Reversion of the Series $a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \dots$, &c. Or rather by multiplying the Sine

$$ra - \frac{a^3}{6r} + \frac{a^5}{120r^3} - \dots, \text{ \&c.}$$

$$\frac{ra - \frac{a^3}{6r} + \frac{a^5}{120r^3} - \dots}{r - \frac{aa}{2r} + \frac{a^4}{24r^3} - \dots} = \text{Tangent.}$$

Cor. 2. The Cotangent of an Arch a is $= \frac{rr}{a} - \frac{a}{3} - \frac{a^3}{45r^2} - \frac{2a^5}{945r^4} - \frac{a^7}{4725r^6} - \frac{2a^9}{9555r^8} - \frac{1382a^{11}}{638512875r^{10}} - \frac{4a^{13}}{18243225r^{12}} - \dots$, &c. as will appear by dividing Ra-

dius Square by the Value of the Tangent in Cor. 1. Or the Cosine by the Sine, and then multiplying by Radius.

Cor. 3. Hence also if τ be the Cotangent of an Arch, then the Arch $= \frac{rr}{\tau} - \frac{\tau^4}{3r^3} + \frac{\tau^6}{5r^5} - \frac{\tau^8}{7r^7} + \frac{\tau^{10}}{9r^9} - \dots$

$$\text{\&c. For } t = \frac{rr}{\tau}$$

PROP.

PROP. XVI.

The Secant CT of an Arch DB being given; to find the Arch. 6.

Let $CD=r$, $CT=f$, $nt=f$, $DT=\sqrt{ff-rr}$, $DB=z$, then by similar Triangles $DT:r::f:nT=\frac{rf}{DT}$, and $f:r::nT$ or $\frac{rf}{DT}:z=\frac{rrf}{f \times DT}$ or $z=\frac{rrf}{f \sqrt{ff-rr}}$, and the Fluent is $z=-\frac{rr}{f}-\frac{r^4}{2.3f^3}-\frac{3r^6}{2.4.5f^5}-\frac{3.5r^8}{2.4.6.7f^7}$, &c. but in D, $z=0$, and $f=r$, therefore the Fluent corrected is, Arch DB or $z=r \times \frac{f-r}{f} + \frac{f^3-r^3}{2.3f^3} + \frac{3\sqrt{f^3-r^3}}{2.4.5f^5} + \frac{3.5\sqrt{f^7-r^7}}{2.4.6.7f^7} + \&c.$

Or thus, in the Point F, $z=a$ Quadrant Q, and $f=\infty$. Therefore the whole Series $=0$. Therefore by Correction $z-Q=0-\frac{rr}{f}-\frac{r^4}{2.3f^3}$, &c. whence Arch z or DB $=Q-\frac{rr}{f}-\frac{r^4}{2.3f^3}-\frac{3r^6}{2.4.5f^5}-\frac{3.5r^8}{2.4.6.7f^7}-\&c.$

Cor. 1. If σ be the Cosecant of an Arch, then the Arch is $=\frac{rr}{\sigma} + \frac{r^4}{2.3\sigma^3} + \frac{3r^6}{2.4.5\sigma^5} + \frac{3.5r^8}{2.4.6.7\sigma^7} + \&c.$ for this is the Complement of the Arch whose Secant is σ , by this Prop.

Cor. 2. In any Arch a , the Secant $=r + \frac{aa}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} + \frac{50521a^{10}}{3628800r^9} + \frac{540553a^{12}}{95800320r^{11}} + \&c.$
For Radius Square (rr), divided by the Cosine ($r-\frac{aa}{2} + \frac{a^4}{24r^3}$, &c.) is = Secant.

Cor.

Cor. 3. The Cofecant of an Arch a is $= \frac{rr}{a} + \frac{a}{6} +$
 $\frac{7a^3}{360r^2} + \frac{31a^5}{15120r^4} + \frac{127a^7}{604800r^6} + \frac{73a^9}{3421440r^8} +$
 $\frac{1414477a^{11}}{653837184000r^{10}} + \&c.$ for Radius Square divided
 by the Sine = Cofecant.

SCHOLIUM.

If an Arch be given in Degrees, its length may be found thus: When the Radius of a Circle is 1, half the Circumference is 3.14159265358979, therefore $\frac{3.14159 \&c.}{180} = .01745329251994 =$ length of 1 Degree. Therefore if r be the Radius of any Circle, then $r \times .01745 \&c. =$ length of the Arch of 1 Degree in that Circle. Consequently $r \times .01745329 \&c. \times$ Number of Degrees and decimal Parts, gives the length of the Arch. And this must be taken for, a or z in the foregoing Propositions.



S E C T. III.

The Calculation of logarithmic Sines, Tangents, and Secants.

Tho' the finding the logarithmic Sines, Tangents, &c. is no more than finding the Logarithms of the natural ones by a Table of Logarithms: Yet these logarithmic Sines, Tangents, &c. may be found without the Table of Logarithms, or else without the natural Sines or Tangents, and sometimes without either, by having only the Arch given.

The logarithmic or artificial Sines, Tangents and Secants are calculated to the Radius 1 with 10 Cyphers annext, viz. 10000000000; so that the Log. Radius will therefore be 10. But in calculating these Sines, Tangents, or Secants, we can more easily compute them for the Radius 1, and then adding 10 gives the Log. Sine, Tangent, or Secant, to the Radius of the Tables. Therefore

In a Circle whose Radius is 1, the Log. of that Radius is 0, and the Length of an Arch of 1 Degree

$$= \frac{3.1415926536}{180} = .01745329252, \text{ and this Num-}$$

ber multiply'd into any Number of Degrees gives the Length of the Arch of these Degrees, for the Radius 1. and this Length must be used in the following Propositions, when the Degrees are given.

Now if you put $M = .4342944819$, and having any Quantity given (expressing a Sine, Tangent, &c.) you have no more to do but to divide the Fluxion of that Quantity by the Quantity itself, and find the Fluent by infinite Series, which multiply'd by M is the Logarithm of the Quantity required.

PROP. XVII.

To find the logarithmic or artificial Sine of the Arch x .

Let $y = \text{nat. Sine of } x$, then by Prop. XII. $y = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$, &c. where $r=1$. And $\dot{y} = \dot{x} - \frac{x^2 \dot{x}}{2} + \frac{x^4 \dot{x}}{24} - \frac{x^6 \dot{x}}{720}$, &c. then by Division $\frac{\dot{y}}{y} = \frac{\dot{x}}{x} - \frac{x \dot{x}}{3} + \frac{x^3 \dot{x}}{45} - \frac{2x^5 \dot{x}}{945} + \frac{x^7 \dot{x}}{4725}$, &c. and Fluent of $M \frac{\dot{y}}{y}$, or the Log. y
 $= \text{Log. } x - M \times : \frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \frac{x^8}{37800} + \text{&c. to}$
 which add 10, and you have the logarithmic Sine in the Tables; and that without knowing the natural Sine.

Otherwise.

Let $z = \frac{1-y}{1+y}$, where $y = \text{nat. Sine}$; then will $y = \frac{1-z}{1+z}$, and $\frac{\dot{y}}{y} = \frac{-2\dot{z}}{1-zz} = -2 \times : \dot{z} + z^2 \dot{z} + z^4 \dot{z} + z^6 \dot{z}$, &c. whence Fl. $M \frac{\dot{y}}{y}$, or Log. $y = -2 M \times z + \frac{1}{3} z^3 + \frac{1}{5} z^5 + \frac{1}{7} z^7$, &c. And the log. Sine of the Tables $= 10 - 2 M \times : z + \frac{1}{3} z^3 + \frac{1}{5} z^5 + \frac{1}{7} z^7$, &c. which is had without the Table of Logarithms.

Or thus.

Let $y = \text{nat. Sine}$, $z = \text{Cofine}$. Then $y = \sqrt{1-zz}$ and $\frac{\dot{y}}{y} = \frac{-z\dot{z}}{1-zz} = -z\dot{z} - z^3\dot{z} - z^5\dot{z}$, &c. whence Fl. $M \frac{\dot{y}}{y} = -\frac{1}{2} M \times : z^2 + \frac{z^4}{2} + \frac{z^6}{3} + \frac{z^8}{4}$, &c. and log. Sine of the Tables $= 10 - \frac{1}{2} M \times : z^2 + \frac{1}{2} z^4 + \frac{1}{3} z^6 + \frac{1}{4} z^8$, &c.

And thus if you have any other Line or Quantity in the Circle, given, by which you can compute the Sine

Sine, you may, from that, find the log. Sine. And FIG. the like may be done for the Tangent, or Secant.

Cor. Hence if $s = \log.$ Sine of $\frac{1}{2} A$, half an Arch given; then the Log. versed Sine of $A = .3010300 + 2s - 10$, for the Tables. For Rad. $\times$ versed Sine = 2 Sine Square of the half Arch, by Cor. 2. Prop. II. and $.3010300 = \log. 2$.

Or thus. Log. vers. of Arch $x = 10.3010300 + 2 \log.$
 $\frac{1}{2} x - 2 M \times : \frac{x^2}{4.6} + \frac{x^4}{4^2.180} + \frac{x^6}{4^3.2835} + \frac{x^8}{4^4.37800} \mathcal{E}c.$
 for s (or the log. Sine of $\frac{1}{2} x$) $= 10 + \log. \frac{1}{2} x$, $- M \times$
 $\frac{x^2}{4.6} + \frac{x^4}{16.180} \mathcal{E}c.$

Or lastly, If $z = \text{Cosine}$, then $\log. \text{vers.} = 10 - M \times$
 $: z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} \mathcal{E}c.$ For $1 - z = \text{versed Sine}$,
 and its Fluxion $\frac{-\dot{z}}{1 - z} = -\dot{z} - z\dot{z} - z^2\dot{z} \mathcal{E}c.$ whose
 Fluent is $-M \times : z + \frac{1}{2}z^2 + \frac{1}{3}z^3 \mathcal{E}c.$ to which add 10.

P R O P. XVIII.

The Arch x being given to find the log. Cosine.

By Cor. 1. Pr. XII. the Cosine $y = 1 - \frac{xx}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
 $\mathcal{E}c.$ therefore $\frac{\dot{y}}{y} = -x\dot{x} - \frac{1}{3}x^3\dot{x} - \frac{2}{15}x^5\dot{x} - \frac{17}{315}x^7\dot{x} \mathcal{E}c.$
 and the Fl. $\frac{M\dot{y}}{y}$ or $\log. y = -M \times : \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6$
 $+ \frac{17}{2520}x^8 \mathcal{E}c.$ And the tab. Cos. of the Arch x is =
 $10 - M \times : \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{17}{2520}x^8 + \mathcal{E}c.$ And that
 without either the nat. Cosine or the Table of Logarithms.

Or the log. Cosine may be found in the same Manner as the Sine, in the two last Methods in Pr. XVII.

PROP. XIX.

Given an Arch z , and the Sine and Cosine of an Arch A ; to find the log. Sine and Cosine of the Sum $A+z$, and also of the Difference $A-z$.

Let the nat. Sine and Cosine of A be s, c . $y = \overline{S.A+z}$. Then by Prop. XIII. $y = s + \frac{cz}{1} - \frac{s^2 z^2}{2} - \frac{cz^3}{6} + \frac{s^2 z^4}{24} \mathcal{E}c$. and $\frac{\dot{y}}{y} = \frac{c\dot{z}}{s} - \frac{z\dot{z}}{ss} + \frac{cz^2\dot{z}}{s^3} - \frac{1+2cc}{3s^4} z^3\dot{z}$, $\mathcal{E}c$. and the Fl. $M \frac{\dot{y}}{y}$ or Log. $y = M \times : \frac{cz}{s} - \frac{z^2}{2ss} + \frac{cz^3}{3s^3} - \frac{1+2cc}{12s^4} z^4$, $\mathcal{E}c$. but when $z=0$, $y=s$, therefore by Correction,

$$\text{Log. } y - \text{Log. } s = M \times : \frac{cz}{s} - \frac{z^2}{2ss} \mathcal{E}c. \text{ whence the}$$

$$\text{Log. } \overline{S.A+z} = \text{Log. } s + M \times : \frac{cz}{s} - \frac{z^2}{2ss} + \frac{cz^3}{3s^3} - \frac{1+2cc}{12s^4} z^4, \mathcal{E}c. \text{ and after the the same manner}$$

$$\text{Log. } \overline{S.A-z} = \text{Log. } s - M \times : \frac{cz}{s} + \frac{z^2}{2ss} + \frac{cz^3}{3s^3} + \frac{1+2cc}{12s^4} z^4, \mathcal{E}c.$$

$$\text{Log. } \overline{\text{Cof. } A+z} = \text{Log. } c - M \times : \frac{sz}{c} + \frac{z^2}{2cc} + \frac{sz^3}{3c^3} + \frac{1+2ss}{12cc} z^4, \mathcal{E}c.$$

$$\text{Log. } \overline{\text{Cof. } A-z} = \text{Log. } c + M \times : \frac{sz}{c} - \frac{z^2}{2cc} + \frac{sz^3}{3c^3} - \frac{1+2ss}{12c^4} z^4, \mathcal{E}c.$$

PROP.

PROP. XX.

The Arch x being given to find its Log. Tangent.

By Cor. 1. Pr. XV. the Tangent $t = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7$, &c. therefore $\frac{t}{x} = 1 + \frac{1}{3}x^2 + \frac{2}{45}x^4 + \frac{17}{945}x^6$, &c. whence Log. Tan. $t = \text{Log. } x + M \times \left(\frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 \right)$, &c. to which add 10.

Otherwise.

To find the Log. Tan. of the Arch $A+x$, the Tan. of the Arch A being given.

Let Tan. $A = a$, Tan. $x = t$, $y = \text{Tan. } \overline{A+x}$. Then by Prop. VIII. $y = \frac{a+t}{1-at}$, and $\dot{y} = \frac{1-aa}{1-at^2} \dot{t}$, and $\frac{\dot{y}}{y} =$

$\frac{1+aa \cdot \dot{t}}{a+1-aa \cdot t-att}$. Let $b = \frac{1+aa}{a}$, $c = \frac{1-aa}{a}$, then $\frac{\dot{y}}{y} = b \times \left(\dot{t} - ctt + \frac{1+cc}{3} t^3 - \frac{2c+c^3}{4} t^5 \right)$, &c. and Fl. $M \dot{y}$ or Log. $y = b M \times \left(t - \frac{1}{2} ctt + \frac{1+cc}{3} t^3 - \frac{2c+c^3}{4} t^5 \right)$, &c. but when x and $t = 0$, $y = a$, then by Correction Log. $y - \text{Log. } a = b M \times \left(t - \frac{1}{2} ctt \right)$, &c. that is

Log. Tan. $\overline{A+x} = \text{Log. Tan. } A + b M \times \left(t - \frac{1}{2} ctt + \frac{1+cc}{3} t^3 - \frac{2c+c^3}{4} t^5 \right)$, &c. and the same for the Log. Tan. $A-x$, changing the Sines of the odd Powers of t .

But if $A = 45^\circ$, then $a = 1$, and $\frac{\dot{y}}{y} = \frac{2\dot{t}}{1-t^2} = 2\dot{t} + 2t^2\dot{t} + 2t^4\dot{t}$, &c. And consequently Log. y or Fl. $M \dot{y} = 2 M \times \left(\dot{t} + \frac{1}{3} t^3 + \frac{1}{5} t^5 \right)$, &c. that is

$$\text{Log. Tan. } \overline{45+x} = 2Mx : t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7}, \text{ \&cc.}$$

But if instead of t you put its Value $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$,
 $\text{\&cc.}$ and the Fluxion for the Fluxion, then will $\frac{2t}{1-tt}$

$$= 2x : \dot{x} + 2x^3\dot{x} + \frac{10}{3}x^4\dot{x} + \frac{244}{45}x^6\dot{x} \text{ \&cc. whence Log. } y,$$

$$\text{or Log. Tan. } \overline{45+x} = 2Mx : x + \frac{2}{3}x^3 + \frac{2}{3}x^5 + \frac{244}{3^{15}}x^7 \text{ \&cc. and likewise}$$

$$\text{Log. Tan. } \overline{45-x} = -2Mx : t + \frac{t^3}{3} + \frac{t^5}{5} \text{ \&cc.} = -2M$$

$x : x + \frac{2}{3}x^3 + \frac{2}{3}x^5$, \&cc. to each of which add 10 for the
 tabular Log. Tang.

$$\text{Cor. The Log. Cot. of } x = 10 - \text{Log. } x - Mx : \frac{x^2}{3} + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 \text{ \&cc.}$$

And Log. Cot. $\overline{A+x}$ is the same as the Log. Tan. of $\overline{A-x}$, found before.

P R O P. XXI.

To find the logarithmic Secant of a given Arch.

Let x be the Arch, y its Secant; then by Cor. 2.

$$\text{Prop. XVI. } y = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6, \text{ \&cc. whence}$$

$$\dot{y} = x\dot{x} + \frac{1}{3}x^3\dot{x} + \frac{2}{15}x^5\dot{x} + \frac{67}{3^{15}}x^7\dot{x}, \text{ \&cc. then Log. } y \text{ or}$$

$$\text{Fluent } M\frac{\dot{y}}{y} = Mx : \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31}{14175}x^{10}$$

$$\text{\&cc. hence the tab. log. Secant} = 10 + Mx : \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} +$$

$$\frac{17x^8}{2520} \text{ \&cc.}$$

Or

Or thus.

To find the Log. Secant of $45+z$. let Log. Secant $45=s$, by Prop. XIX. $\log. \text{Cof. } 45+z = \log. \text{Cof. } 45 - M \times \frac{s^2}{c} + \frac{z^2}{2cc}$, $\&c.$ (because $s=c$, $r=1$.)
 $\log. \text{Cof. } 45 - M \times : z+z^2$, $\&c.$ but by Cor. 2. Pr. I. Secant $\times$ Cosine $=1$. Or $\log. \text{Secant} = -\log. \text{Cosine}$. Therefore, $\log. \text{Sec. } 45+z = \log. \text{Sec. } 45 + M \times : s^2+z^2$, $\&c.$ that is

Log. Sec. $45+z = s + M \times : z+z^2+\frac{2}{3}z^3+\frac{1}{5}z^5$, $\&c.$

Cor. The log. Cosecant of an Arch x is $=10 - \log. x + M \times : \frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \frac{x^8}{37800}$, $\&c.$ For $\log. \text{Cosecant} = -\log. \text{Sine}$, when Radius is 1.

Or $\log. \text{Cosecant of } 45+z$, or $\log. \text{Secant of } 45-z = s - M \times : z - z^2 + \frac{2}{3}z^3 - \frac{1}{5}z^5$, $\&c.$ by this Prop. changing the Signs of the odd Powers of z .

SCHOLIUM I.

Having the log. Sine, Tangent or Secant giver, the Arch belonging thereto may be found by the Reversion of Series. As suppose there is given T the $\log. \text{Tan. } 45+x$, then $T = 10 + 2M \times : x + \frac{2}{3}x^3 + \frac{2}{5}x^5$, $\&c.$ by Pr. XX, and $x + \frac{2}{3}x^3 + \frac{2}{5}x^5$, $\&c. = \frac{T-10}{2M} = l$

suppose, then by Reversion $x = l - \frac{1}{3}l^3 + \frac{14}{15}l^5$, $\&c.$ and

$\frac{x}{.0174532} = \text{Degrees in } x$.

Again, let there be given K, the log. Secant of $45+z$, then by Prop. XXI. $K = s + M \times : z + z^2 + \frac{2}{3}z^3$, $\&c.$ then $z + z^2 + \frac{2}{3}z^3$, $\&c. = \frac{K-s}{M} = l$. Then by Reversion $z = l - ll + \frac{4}{3}l^3$, $\&c.$ which divided by .0174532 will give the Degrees in z .

FIG.

SCHOLIUM II.

In any Arch let $\log.$ Radius $= r = 10.$

$s = \log.$ Sine. $t = \log.$ Tangent. $f = \log.$ Secant.
 $c = \log.$ Cofine. $\tau = \log.$ Cotangent. $\sigma = \log.$ Cofecant.

Then

$$s = c + t - r = t + r - f = c + r - \tau = 2r - \sigma.$$

$$c = s + r - t = 2r - f = s + \tau - r = \tau + r - \sigma.$$

$$t = s + r - c = 2r - \tau.$$

$$\tau = c + r - s = 2r - t.$$

$$f = t + r - s = 3r - s - \tau = \sigma + r - \tau = 2r - c.$$

$$\sigma = \tau + r - c = 3r - c - t = f + r - t = 2r - s.$$

By what has been delivered in the two last Sections, the Sines, Tangents and Secants, of all Arches, whether natural or artificial, may easily be found; and from thence the Table of Sines and Tangents may be constructed, with great Ease and Expedition. Or any particular Numbers in the Tables may be computed anew, and any Errors corrected therein. But the nat. Sines and Tangents being first found, the artificial ones are most easily had by a Table of Logarithms; for this Reason I have not continu'd the several Series in this last Section to any great Number of Places; intending rather to give the Reader the Principles of Calculation than the Calculation itself; since in all Probability there will be few Persons who will take the Pains to calculate them anew. But if any has a Mind to do it, he may himself continue any of these Series as far as he pleases.

SECT.

S E C T. IV.

*The Calculation of the Sines, Cosines, Cords, &c.
of multiple Arches.*

P R O P. XXII.

If a Trapezoid BCDE, whose Sides BE, CD are parallel, be inscribed in a Circle; and the Cords BD, CE be drawn; I say, $BD^2 - DE^2 = BE \times CD$. FIG. 19.

Make Angle $EBO = CBD$, then $EBD = CBO$; and since $BEO = CBD$, therefore the Triangles BCD and BOE are similar, and $BD : CD :: BE : OE$, and $BD \times OE = BE \times CD$. Also since Angle $BCO = BDE$, the Triangles BCO, and BDE are similar, whence $BD : DE :: BC : CO$, and $BD \times CO = DE \times BC = DE^2$. Therefore $BD \times OE + OC = BE \times CD + DE^2$, that is $BD \times CE$ or $BD^2 = BE \times CD + DE^2$, or $BE \times CD = BD^2 - DE^2 = \overline{BD + DE} \times \overline{BD - DE}$.

Cor. 1. If A be any Arch, and BC or DE = n times A, and BD = m times A. Then

As Cord of $m - n$. A :
Cord . m A + Cord . n A ::
Cord . m A - Cord . n A :
Cord . $m + n$. A.

For CD is the Cord of $m - n$ times A, and BE of $m + n$ times A.

Cor. 2. As Cord of A :
Cord . $n + 1$. A + Cord . n A ::
Cord . $n + 1$. A - Cord . n A :
Cord . $2n + 1$. A.

This appears by putting $m = n + 1$, or $CD = A$.

Cor.

FIG.

19.

Cor. 3. As Cord of $\overline{m-1} \cdot A$:Cord . $m A +$ Cord . A ::Cord . $m A -$ Cord . A :Cord $\overline{m+1} \cdot A$.This appears by putting $n=1$, or CB or $DE=A$.Cor. 4. If three Arches CD , BCD , $BCDE$ are in arithmetic Progreſſion, $BD^2 - BC^2 = CD \times BE$.

P R O P. XXIII.

22. *In the Circle AEB, whose Diameter is AB, if the Arches AD, EF, FG be taken equal to one another, and the Cords BD, BE, BF, BG drawn. I ſay, As Radius CB : Cord BD :: ſo the middle Cord BF : BE + BG, the Sum of the extreame Cords.*

For produce BE to H, and make $FH = FB$, then $\angle BHF = HBF = FBG$, and draw the Cords EF, FG. Then the Angle $BEF = BFG + FBG$, both ſtanding on the ſame Arch FGB. Alſo $BEF = EFH + EHF = EFH + FBG$, therefore $EFH = BFG$. And the Triangles EFH, FBG are ſimilar and equal. Whence $HE = GB$, and $HB = EB + BG$. Alſo the Triangles DCB and HFB are ſimilar, therefore $BC : BD :: BF : BH$ or $EB + BG$.

Cor. If Arch $AD = A$, Arch $BF = n \times A$. Then As Rad :

Cord . ſup. A ::Cord . $n A$:Cord . $\overline{n-1} \cdot A +$ Cord . $\overline{n+1} \cdot A$.

And the ſame holds in reſpect of the Cords of the Supplements of theſe Arches, if AF be ſuppoſed to be made equal to $n A$.

S C H O L I U M.

If the Cord BG lye on the other Side of B, then BG is negative.

P R O P.

The Sine and Cosine of an Arch, being given; to find the Sine of n times that Arch.

Let $x = \text{Cosine}$, $y = \text{Sine}$, given. And A the given Arch, and at present let Radius $= 1$. Then by Cor. 7. Prop. III. $2x \times S.nA = S.n-1.A + S.n+1.A$, whence $S.n+1.A = 2x \times S.nA - S.n-1.A$. Or because $xx + yy = 1$. $S.n+1.A = 2x \times S.nA - xx + yy \times S.n-1.A$. That is if any Sine be multiply'd by $2x$, and the next preceding one multiply'd by $xx + yy$, and subtracted from it, gives the next following Sine. Thus you will have

$$S. A = y.$$

$$S. 2A = 2xy.$$

$$S. 3A = 3x^2y - y^3.$$

$$S. 4A = 4x^3y - 4xy^3.$$

$$S. 5A = 5x^4y - 10x^2y^3 + y^5. \text{ \&c.}$$

In what follows, if a denotes any variable Quantity, then I denote its first and second succeeding Values by a , a , $\text{\&c.}$ and its first and second preceding Values by a , a , $\text{\&c.}$ and so of others. And with these I proceed according to the Method of Increments in calculating the following Propositions.

Suppose then any Arch, as nA , is represented in general thus, $S.nA = nx^{n-1}y - ax^{n-3}y^3 + bx^{n-5}y^5 - \text{\&c.}$ then according to the Construction of Sines, before mentioned, we shall have

$$\left\{ \begin{array}{l} 2nx^n y - 2ax^{n-2}y^3 + 2bx^{n-4}y^5 - \text{\&c.} = S.nA \times 2x, \\ -nx^n y + ax^{n-2}y^3 - bx^{n-4}y^5 - \text{\&c.} \end{array} \right\} = \overline{xx + yy} \times S.n-1.A.$$

$$= \overline{n+1. x^n y - ax^{n-2}y^3 + bx^{n-4}y^5 - \text{\&c.}} = S.n+1.A.$$

Hence equating the Coefficients of the homologous Powers, $n+1 = 2n-n$, and $1 = n-n$. That is $n=1$.

likewise

FIG. likewise $a = 2a - a + n$, or $a = 2a - a + n$, whence $a - 2a + a = n$, that is $a = n$. And the Integral is $a =$

$$\frac{n n}{2}, \text{ and the Integral of this is } a = \frac{n n}{2 \cdot 3} \text{ (for when}$$

$$n=3, a = \frac{n n}{2 \cdot 3} = 1). \text{ Again } b = 2b - b + a, \text{ or } b =$$

$$2b - b + a, \text{ whence } b - 2b + b = a, \text{ or } b = a = \frac{n n}{2 \cdot 3},$$

$$\text{whence } b = \frac{n n n n}{2 \cdot 3 \cdot 4}, \text{ and } b = \frac{n n n n n n}{2 \cdot 3 \cdot 4 \cdot 5}. \text{ In like}$$

$$\text{manner } c = b, \text{ and } c = \frac{n n n n n n n n}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}, \text{ \&c. whence}$$

$$S.n A = n x^{n-1} y - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} x^{n-3} y^3 +$$

$$\frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^{n-5} y^5, \text{ \&c. or putting } r =$$

$$\text{Rad. } S.n A = \frac{n x^{n-1} y}{r^{n-1}} - \frac{n-1 \cdot n-2 \cdot y y}{2 \cdot 3 r r} A - \frac{n-3 \cdot n-4 \cdot y^2}{4 \cdot 5 r r} B$$

$$- \frac{n-5 \cdot n-6 \cdot y y}{6 \cdot 7 r r} C, \text{ \&c. where } A, B, C, \text{ \&c. are the}$$

preceding Terms with their Signs.

Cor. If you suppose r to be the Diameter, y the Cord of an Arch, x the Cord of the Supplement; then the foregoing Series will be the Cord of the multiple Arch.

P R O P. XXV.

The Sine s of an Arch A , being given; to find the Sine of $n A$, or n times that Arch.

Let $z = S.n A$. Then by Prop. XI. the Arch belonging

longing to the Sine s , is $s + \frac{s^3}{6rr} + \frac{3s^5}{40r^4} + \frac{5s^7}{112r^6}$, &c.

and the Arch belonging to the Sine z is $z + \frac{z^3}{6rr} +$

$\frac{3z^5}{40r^4} + \frac{5z^7}{112r^6}$, &c. and this is n times the former,

therefore $z + \frac{z^3}{6r^2} + \frac{3z^5}{40r^4} + \frac{5z^7}{112r^6}$, &c. $= ns + \frac{ns^3}{6rr} +$

$\frac{3ns^5}{40r^4} + \frac{5ns^7}{112r^6}$, &c.

Now suppose $z = as + \frac{bs^3}{6rr} + \frac{cs^5}{40r^4} + \frac{ds^7}{112r^6}$, &c.

then will $\frac{z^3}{6rr} = \frac{a^3s^3}{6rr} + \frac{aab}{2rr}s^5 + \frac{abb+aac}{2rr}s^7$, &c.

$\frac{3z^5}{40r^4} = \frac{3a^5}{40r^4}s^5 + \frac{3a^4b}{8r^4}s^7$, &c.

$\frac{5z^7}{112r^6} = \frac{5a^7}{112r^6}s^7$, &c.

$ns + \frac{n}{6rr}s^3 + \frac{3ns^5}{40r^4} + \frac{5n}{112r^6}s^7$, &c. then by equating

the homologous Powers of s , we have $as = ns$, and

$a = n$. Also $b + \frac{a^3}{6rr} = \frac{a}{6rr}$, and thence $b = \frac{n - n^3}{6rr} =$

$\frac{n \cdot 1 - nn}{2 \cdot 3 \cdot rr}$. Also $c + \frac{aab}{2rr} + \frac{3a^5}{40r^4} = \frac{3n}{40r^4}$, whence $c =$

$\frac{9n - 10n^3 + n^5}{120r^4} = \frac{n \cdot 1 - n^2 \cdot 9 - nn}{2 \cdot 3 \cdot 4 \cdot 5 r^4}$. In like manner $d =$

$\frac{225n - 259n^3 + 35n^5 - n^7}{5040r^6} = \frac{n \cdot 1 - nn \cdot 9 - nn \cdot 25 - nn}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 r^6}$, &c.

whence $S.nA = ns - \frac{n \cdot nn - 1}{2 \cdot 3 r^2}s^3 + \frac{n \cdot nn - 1 \cdot nn - 3^2}{2 \cdot 3 \cdot 4 \cdot 5 r^4}s^5 -$

$\frac{n \cdot nn - 1 \cdot nn - 3^2 \cdot nn - 5^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 r^6}s^7$, &c. that is

$S.nA$

FIG.

$$S.nA = ns - \frac{nn-1}{2.3r^2} s^2 A - \frac{nn-3^2}{4.5r^2} s^2 B - \frac{n^2-5^2}{6.7r^2} s^2 C -$$

$\frac{n^2-7^2}{8.9r^2} s^2 D, \text{ \&c.}$ where $A, B, C,$ are the preceding Terms with their Sines. Therefore if n be an odd Number the Series will be finite.

Cor. 1. Since the Cofine of $A = \sqrt{rr-ss} = r - \frac{ss}{2r} - \frac{s^4}{8r^3} - \frac{s^6}{16r^5}, \text{ \&c.}$ therefore dividing the Series above by this Series, and then multiplying by $\sqrt{rr-ss}$, and you will have

$$S.A = \frac{ns}{r} - \frac{n^2-2^2}{2.3r^2} s^2 A - \frac{n^2-4^2}{4.5r^2} s^2 B - \frac{n^2-6^2}{6.7r^2} s^2 C - \frac{n^2-8^2}{8.9r^2} s^2 D, \text{ \&c.} \times \sqrt{rr-ss}.$$

And therefore if n be an even Number the Series will be finite.

Cor. 2. If s be the Cord of an Arch, x the Cord of the Supplement $d =$ Diameter, then

$$\text{Cord of } nA = ns - \frac{nn-1}{2.3dd} ss A - \frac{n^2-3^2}{4.5.d^2} ss B - \frac{n^2-5^2}{6.7.d^2} ss C - \frac{n^2-7^2}{8.9.d^2} ss D, \text{ \&c. or}$$

$$\text{Cord of } n.\text{Arch} = \frac{nsx}{d} - \frac{n^2-2^2}{2.3.d^2} s^2 A - \frac{n^2-4^2}{4.5.d^2} s^2 B - \frac{n^2-6^2}{6.7.d^2} s^2 C - \text{ \&c.}$$

S C H O L I U M.

If z the Sine of an Arch was given, and the Sine s , of the n^{th} Part of the Arch, was required; you will have by Reversion of Series,

$$s = \frac{z}{n} + \frac{n^2-1.z^2}{2.3r^2n^2} A + \frac{3n^2-1}{4.5r^2n^2} Bz^2 + \frac{5n^2-1}{6.7r^2n^2} Cz^2 \text{ \&c.}$$

PROP.

PROP. XXVI.

The Sine and Cosine of an Arch being given; to find the Cosine of n times that Arch.

Let $A = \text{Arch}$, $y = \text{Sine}$, $x = \text{Cosine}$. Radius $= 1$.
Then by Cor. 7. Pr. III. $2x \times \text{Cof. } nA = \text{Cof. } n-1.A + \text{Cof. } n+1.A$, therefore since $xx+yy=1$, $\text{Cof. } n+1.A = 2x \times \text{Cof. } nA - xx+yy \times \text{Cof. } n-1.A$:
Therefore if you multiply any Cosine by $2x$, and from it subtract the foregoing one multiply'd into $xx+yy$, the remainder is the next succeeding Cosine.
Hence

$$\text{Cof. } 0A = r$$

$$\text{Cof. } 1A = x$$

$$\text{Cof. } 2A = xx - yy$$

$$\text{Cof. } 3A = x^3 - 3xyy$$

$$\text{Cof. } 4A = x^4 - 6x^2y^2 + y^4$$

$$\text{Cof. } 5A = x^5 - 10x^3y^2 + 5xy^4$$

$$\text{Cof. } 6A = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$$

$$\text{Cof. } 7A = x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6.$$

Et c. therefore in general, let the

$$\text{Cof. } n-1.A = x^{n-1} - ax^{n-3}y^2 + bx^{n-5}y^4 - cx^{n-7}y^6 \text{ \&c.}$$

$$\text{Cof. } nA = x^n - ax^{n-2}y^2 + bx^{n-4}y^4 - cx^{n-6}y^6 \text{ \&c.}$$

Now if the latter be multiply'd by $2x$, and the former by $xx+yy$, and this Product subtracted from the first, you will have

$$\left. \begin{aligned} &2x^{n+1} - 2ax^{n-1}y^2 + 2bx^{n-3}y^4 - 2cx^{n-5}y^6 \text{ \&c.} \\ &- x^{n+1} + ax^{n-1}y^2 - bx^{n-3}y^4 + cx^{n-5}y^6 \\ &- x^{n-1}y^2 + ax^{n-3}y^4 - bx^{n-5}y^6 \end{aligned} \right\}$$

$$= x^{n+1} - ax^{n-1}y^2 + bx^{n-3}y^4 - cx^{n-5}y^6 \text{ \&c.} = \text{Cof.}$$

$n+1.A$. Then comparing the Coefficients we have $a=2a-a+1$, that is, by the Method of Increments

$a+2a+a=2a+2a-a+1$, or $a=1=n$, therefore $a=$
 n , the

n , the Integral, and $a = \frac{n}{1 \cdot 2}$, for when $n=2$, a or

$\frac{n}{2} = 1$. Likewise $b = 2b - b + a$, and $b = a = \frac{n}{2}$,

and $b = \frac{n \cdot n}{2 \cdot 3}$, and $b = \frac{n \cdot n \cdot n}{2 \cdot 3 \cdot 4}$, (for when $n=5$,

$b=1$.) and $b = \frac{n \cdot n \cdot n \cdot n}{1 \cdot 2 \cdot 3 \cdot 4}$. Again $c = b = \frac{n \cdot \dots \cdot n}{2 \cdot 3 \cdot 4}$, $c =$

$\frac{n \cdot \dots \cdot n}{2 \cdot 3 \cdot 4 \cdot 5}$ and $c = \frac{n \cdot \dots \cdot n}{2 \cdot \dots \cdot 6}$ and therefore $c = \frac{n \cdot \dots \cdot n}{1 \cdot 2 \cdot \dots \cdot 6}$.

Likewise $d = \frac{n \cdot \dots \cdot n}{1 \cdot 2 \cdot \dots \cdot 7}$, &c. then since $n=1$, therefore n , n , n , &c. are equal to n , $n-1$, $n-2$, &c. whence putting r for Radius,

$$\text{Cos. } nA = \frac{1}{r^{n-1}} \times : x^n - \frac{n \cdot n-1}{2} x^{n-2} y^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{2 \cdot 3 \cdot 4} x^{n-4} y^4 - \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4 \cdot n-5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} x^{n-6} y^6$$

&c. or

$$\text{Cos. } nA = \frac{x^n}{r^{n-1}} - \frac{n \cdot n-1}{1 \cdot 2} A \frac{yy}{xx} - \frac{n-2 \cdot n-3}{3 \cdot 4} B \frac{yy}{xx} - \frac{n-4 \cdot n-5}{5 \cdot 6} C \frac{yy}{xx} - \frac{n-6 \cdot n-7}{7 \cdot 8} D \frac{yy}{xx}, \text{ \&c. where } A, B, C, \text{ \&c. are the foregoing Terms with their Signs.}$$

Cor. If you suppose r the Diameter, y the Cord of an Arch, x the Cord of the Supplement, then the foregoing Series will be the Cord of the Supplement of the multiple Arch.

PROP.

PROP. XXVII.

The Sine s of an Arch a being given; to find the Cosine of na , or n times that Arch.

Let $z = \text{Cof. } na, \text{ Rad.} = 1$, then by Cor. 1. Pr. 12.

$$z = 1 - \frac{n^2 a^2}{2} + \frac{n^4 a^4}{24} - \frac{n^6 a^6}{720} + \frac{n^8 a^8}{40320} \text{ \&c. but by}$$

$$\text{Prop. 11. } a = s + \frac{s^3}{6} + \frac{3s^5}{40} + \frac{5s^7}{112} \text{ \&c.}$$

$$\text{whence } aa = ss + \frac{s^4}{3} + \frac{8s^6}{45} + \frac{4s^8}{35} \text{ \&c.}$$

$$\text{and } a^4 = s^4 + \frac{2}{3}s^6 + \frac{7}{15}s^8 \text{ \&c.}$$

$$a^6 = s^6 + s^8$$

$$a^8 = s^8$$

$$\text{\&c.}$$

then all these Values of the Powers of a being substituted in the Value of z , gives

$$\begin{aligned} z = 1 - \frac{nnss}{2} - \frac{n^2s^4}{6} - \frac{4n^2s^6}{45} - \frac{2n^2s^8}{35} \text{ \&c.} \\ + \frac{n^4s^4}{24} + \frac{n^4s^6}{36} + \frac{7n^4s^8}{360} \\ - \frac{n^6s^6}{720} - \frac{n^6s^8}{720} \\ + \frac{n^8s^8}{40320} \text{ \&c.} \end{aligned}$$

whence

$$z = 1 - \frac{n^2}{2} s^2 + \frac{nn \cdot n^2 - 2^2}{2 \cdot 3 \cdot 4} s^4 - \frac{n^2 \cdot n^2 - 2^2 \cdot n^2 - 4^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} s^6 \text{ \&c.}$$

that is if $r = \text{Radius}$.

$$\text{Cof. } na = r - \frac{n^2 s^2}{1 \cdot 2 r^2} A - \frac{n^2 - 2^2}{3 \cdot 4 r^2} s s B - \frac{n^2 - 4^2}{5 \cdot 6 r^2} s s C - \frac{n^2 - 6^2}{7 \cdot 8 r^2} s s D \text{ \&c. where A, B, C, \&c. are the foregoing Terms with their Signs. Therefore if } n \text{ be an even Number, the Series will be finite.}$$

E

Cor.

Cor. Since $\sqrt{rr-ss}=r-\frac{ss}{2r}-\frac{s^4}{8r^3}-\frac{s^6}{16r^5}-\frac{5s^8}{128r^7}\text{ \&c.}$
 $=$ Cof. A. therefore dividing the foregoing Series, by
 this; and then multiply by $\sqrt{rr-ss}$, and you will
 have Cof. $na=\sqrt{rr-ss} \times : 1-\frac{n^2-1}{1.2r^2}ssA-\frac{n^2-3^2}{3.4r^2}ssB$
 $-\frac{n^2-5^2}{5.6r^2}ssC-\frac{n^2-7^2}{7.8r^2}ssD\text{ \&c.}$ and therefore if n be
 an odd Number the Series will be finite.

P R O P. XXVIII.

*The Cord of an Arch, and the Cord of its Supplement
 being given; to find the Cord of n times that Arch.*

Let A be the Arch, y the Cord, x the Cord of the
 Supplement, Radius $=1$, then by *Cor. Pr. XXIII.*
 $x \times \text{Cord. } nA = \text{Cord. } \overline{n-1}. A + \text{Cord. } \overline{n+1}. A$, and
 $\text{Cord. } \overline{n+1}. A = x \times \text{Cord. } nA - \text{Cord. } \overline{n-1}. A$. that is,
 if any Cord be multiply'd by x , and the foregoing
 one subtracted, will give the next following Cord.
 whence

$$\begin{aligned}\text{Cord. } A &= y \\ \text{Cord. } 2A &= xy \\ \text{Cord. } 3A &= x^2y - y \\ \text{Cord. } 4A &= x^3y - 2xy \\ \text{Cord. } 5A &= x^4y - 3x^2y + y, \text{ \&c.}\end{aligned}$$

And in general let the Cord of, nA be $= yx^{n-1} -$
 $ax^{n-3}y + bx^{n-5}y - cx^{n-7}y \text{ \&c.}$ then will

$$\left. \begin{aligned}yx^n - ayx^{n-2} + byx^{n-4} - cyx^{n-6} \text{ \&c.} \\ - yx^{n-2} + ayx^{n-4} - byx^{n-6} \text{ \&c.}\end{aligned} \right\} =$$

$$yx^n - ayx^{n-2} + byx^{n-4} - cyx^{n-6} \text{ \&c.} = \text{Cord. } \overline{n+1}. A.$$

then equating the Coefficients $a=a+1$, and $a-a$ or
 $a=1=n$, and the Integral is $a=n$ (for when $n=3$, a
 -2

or

or $n=1$). Again, b or $b+b=b+a$, or $b=a=n$, and FIG.

$$b = \frac{n}{2} \frac{n-1}{2}, \text{ also } c = c+b, \text{ or } c = b = \frac{n-1}{2} \frac{n-2}{2}, \text{ and } c = \frac{n}{2} \frac{n-1}{2} \frac{n-2}{2} \frac{n-3}{2}.$$

In like manner $d = \frac{n}{2} \frac{n-1}{2} \frac{n-2}{2} \frac{n-3}{2} \frac{n-4}{2}$, &c. whence

$$\text{the Cord } nA = yx^{n-1} - \frac{n-2}{1} yx^{n-3} + \frac{n-3}{1 \cdot 2} yx^{n-5} - \frac{n-4}{1 \cdot 2 \cdot 3} yx^{n-7} + \frac{n-5}{1 \cdot 2 \cdot 3 \cdot 4} yx^{n-9} \text{ &c.}$$

that is putting Radius $=r$, then

$$\text{Cord } n \text{ Arch} = \frac{y x^{n-1}}{r^{n-1}} - \frac{n-2}{1} A \frac{r r}{x x} - \frac{n-3}{2 \cdot n-2} B \frac{r r}{x x} - \frac{n-4}{3 \cdot n-3} C \frac{r r}{x x} - \frac{n-5}{4 \cdot n-4} D \frac{r r}{x x} - \frac{n-6}{5 \cdot n-5} E \frac{r r}{x x} \text{ &c.}$$

Cor. If $r = \text{Radius}$, $y = \text{Sine}$, $x = \text{Cofine}$ of an Arch,

$$\text{then the Sine of the } n^{\text{th}} \text{ Arch} = \frac{2x^{n-1}}{r^{n-1}} y - \frac{n-2}{4xx} A - \frac{n-3}{4 \cdot 2 \cdot n-2} B - \frac{n-4}{4 \cdot 3 \cdot n-3} C - \frac{n-5}{4 \cdot 4 \cdot n-4} D \text{ &c.}$$

this is plain by putting $\frac{1}{2}r$ for r in the foregoing Series.

P R O P. XXIX.

The Square of the Cord of an Arch being given; to find the Square of the Cord of n times the Arch.

Let a, b, c be the Cords of three Arches in arithmetic Progression, Rad. $=1$, $x = \text{Cord of the common Difference}$, and $\sqrt{4-xx} = \text{Cord of the Sup. common Difference}$. Then by Prop. XXIII. Rad. $\times \text{Sum of the Cords of the extream Arches} = \text{Cord. mean} \times \text{Cord Sup. common Difference}$, that is, $b\sqrt{4-xx} = a+c$. And by Cor. 4. Pr. XXII. $ac = bb - xx$

FIG. $-xx$. Whence $4bb - bbxx = aa + cc + 2ac = aa + cc + 2bb - 2xx$, therefore $bb + 2 - xx + 2xx - aa = cc$, that is, in any Series of the Squares of Cords, if the Square of any Cord be multiply'd by $-xx + 2$, and the preceding one subtracted, and $2xx$ added, you will have the Square of the next following Cord. Thus, Square of the Cord

of $A = xx$

$$2A = -x^4 + 4x^2$$

$$3A = x^6 - 6x^4 + 9x^2$$

$$4A = -x^8 + 8x^6 - 20x^4 + 16x^2$$

$$5A = x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2$$

$\&c.$

therefore let any one in general be express'd thus,

$$x^v - ax^{v-2} + bx^{v-4} - cx^{v-6} \&c.$$

then the next succeeding one, (changing the Signs),

$$\text{will be } -x^{v-1} + ax^{v-2} - bx^{v-4} + cx^{v-6} \&c. = (\text{by}$$

Construction to)

$$\left. \begin{array}{l} -x^{v+2} + ax^v \\ + 2x^v \\ - bx^{v-2} + cx^{v-4} \\ - 2ax^{v-2} + 2bx^{v-4} \\ + x^{v-2} - ax^{v-4} \end{array} \right\} \&c. + 2xx.$$

then comparing the homologous Terms we have,

$v = v + 2$, or $v = v$ that is $v = 2 = 2n$, and $v = 2n$. Again,

$a = a + 2$, or $a = a$ or $a = 2 = v$, whence $a = v = 2n$.

Also $b = b + 2a - 1$, or $b = 2a - 1 = 2v - 1 = 2v - \frac{1}{2}v = \frac{3}{2}v$,

$$\text{therefore } b = \frac{v}{2} - \frac{v}{2}.$$

$$\text{Again, } c = c + 2b - a, \text{ or } c = v - v - v = v - 2v + 2 =$$

$$-v - 2v + v, \text{ and } c = \frac{-2 - 1}{2 \cdot 3} - \frac{-1}{2} + v.$$

In

In like manner $d = 2c - b = \frac{-2}{-1} - \frac{-1}{3} - \frac{3}{2} + \frac{9}{2} - \frac{5}{2}$

and $d = \frac{-3}{2} - \frac{-1}{3} - \frac{-1}{2} + \frac{9}{2} - \frac{5}{2}$

But since $v = 2$. $v = v - 2$. $v = v - 4$ &c. we shall find

$b = v \times \frac{v-3}{2}$, $c = v \times \frac{v-4}{2} \times \frac{v-5}{3}$, $d = v \times \frac{v-5 \times v-6 \times v-7}{2 \cdot 3 \cdot 4}$

&c. therefore if x is the Cord of an Arch, and b the Cord of n times the Arch, then will

$x^{2n} - 2nx^{2n-2} + 2n \times \frac{2n-3}{2} x^{2n-4} - 2n \times \frac{2n-4 \cdot 2n-5}{2 \cdot 3} x^{2n-6}$
 $+ 2n \times \frac{2n-5 \cdot 2n-6 \cdot 2n-7}{2 \cdot 3 \cdot 4} x^{2n-8}$ &c. = $\pm bb$, accord-

ing as n is odd or even; and the Series continued to n Terms.

Or $\frac{x^{2n}}{r^{2n-1}} - \frac{2nr^2}{xx} A - \frac{2n-3}{2xx} rr B - \frac{2n-4 \cdot 2n-5}{3 \cdot 2n-3 \cdot xx} rr C$
 $- \frac{2n-6 \cdot 2n-7}{4 \cdot 2n-4 \cdot xx} rr D - \frac{2n-8 \cdot 2n-9}{5 \cdot 2n-5 \cdot xx} rr E -$
 $\frac{2n-10 \cdot 2n-11}{6 \cdot 2n-6 \cdot xx} rr F -$ &c. = $\pm bb$.

Otherwise.

Since the Law of the Continuation of the Series of the Squares of the Cords is this, that any Cord multiply'd by $2 - xx$, and the foregoing one subtracted, and $2xx$ added, gives the next. Therefore

□ 1 Cord = xx

2. = $4x^2 - x^4$

3. = $9x^2 - 6x^4 + x^6$

4. = $16x^2 - 20x^4 + 8x^6 - x^8$

5. = $25x^2 - 50x^4 + 35x^6 - 10x^8 + x^{10}$
 &c.

E 3

and

FIG. and in general $ax^2 - bx^4 + cx^6 - dx^8 \&c. =$ Square of the Cord of the n^{th} Arch. Whence by the Construction,

$$\left. \begin{array}{cccc} 2ax^2 & -2bx^4 & +2cx^6 & -2dx^8 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ +2 & -a & +b & -c \\ -a & +b & -c & +d \end{array} \right\} =$$

$$\frac{ax^2}{2} - \frac{bx^4}{2} + \frac{cx^6}{2} - \frac{dx^8}{2} \&c.$$

Here then $a = 2a - a + 2$, that is by the Method of Increments $a + 2a + a = 2a + 2a - a + 2$, that is $a = 2 =$

$2n$, and $a = 2n + A$. and $a = \frac{-1}{2} + An$, and when $n = 1$, $A = 1$, therefore $a = nn + n$.

Again, $b = 2b + a - b$, and their Increments $b + 2b + b = 2b + 2b + a + a - b$, and $b = a + a$, likewise $c = b + b$, $d = c + c$ and so on; that is, $b = a$, $c = b$, $d = c \&c.$

therefore $b = nn + n$, and $b = \frac{-1}{3} + \frac{1}{2}$, and $b =$
 $\frac{n}{3 \cdot 4} + \frac{nn}{2 \cdot 3}$. Again, $c = \frac{-1}{3 \cdot 4} + \frac{1^2}{2 \cdot 3}$, and $c =$
 $\frac{n}{3 \cdot 4 \cdot 5} + \frac{nn}{2 \cdot 3 \cdot 4}$, and $c = \frac{-3}{3 \cdot \dots \cdot 6} + \frac{-2}{2 \cdot \dots \cdot 5}$. In like
manner $d = \frac{-4}{3 \cdot \dots \cdot 8} + \frac{-3}{2 \cdot \dots \cdot 7}$, and $e = \frac{-5}{3 \cdot \dots \cdot 10} + \frac{-4}{2 \cdot \dots \cdot 9}$,
and so on. Then since $n = 1$. and n , n , n , n , $n \&c. =$
 n , $n - 1$, $n + 1$, $n - 2$, $n + 2 \&c.$ respectively. There-
fore by Reduction will be found $a = nn$, $b = \frac{nn \cdot nn - 1}{2 \cdot 2 \cdot 3}$,
 $c =$

$$c = \frac{nn \cdot nn - 1 \cdot nn - 4}{3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}, d = \frac{nn \cdot nn - 1 \cdot nn - 4 \cdot nn - 9}{4 \cdot 1 \cdot \dots \cdot 7} \text{ \&c.}$$

whence the Square of the Cord of the n^{th} Arch $= nnx^2 -$

$$\frac{nn \cdot nn - 1}{2 \cdot 1 \cdot 2 \cdot 3} x^4 + \frac{nn \cdot nn - 1 \cdot nn - 4}{3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^6 -$$

$$\frac{nn \cdot nn - 1 \cdot nn - 4 \cdot nn - 9}{4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} x^8 \text{ \&c. or putting } r = \text{Radi-}$$

$$\text{us, } \square \text{ Cord } n^{\text{th}} \text{ Arch} = nnx^2 - \frac{nn - 1 \cdot x^2}{2 \cdot 1 \cdot 2 \cdot 3 r^2} A - \frac{2 \cdot nn - 4 \cdot x^2}{3 \cdot 4 \cdot 5 \cdot rr} B$$

$$- \frac{3 \cdot nn - 3^2}{4 \cdot 6 \cdot 7 r^2} x^2 C - \frac{4 \cdot n^2 - 4^2 \cdot x^2}{5 \cdot 8 \cdot 9 r^2} D - \frac{5 \cdot n^2 - 5^2 \cdot x^2}{6 \cdot 10 \cdot 11 \cdot r^2} E \text{ \&c.}$$

Cor. 1. If x be the Cord of an Arch, Radius $= r$, and c the Cord of n times the Arch; then $\frac{x^n}{r^n - 1} -$

$$\frac{nx^{n-2}}{r^{n-3}} + n \times \frac{n-3 \cdot x^{n-4}}{2r^{n-5}} - n \times \frac{n-4 \cdot n-5 \cdot x^{n-6}}{2 \cdot 3 r^{n-7}} + n \times \frac{n-5 \cdot n-6 \cdot n-7}{2 \cdot 3 \cdot 4 r^{n-9}} x^{n-8} - n \times \frac{n-6 \cdot n-7 \cdot n-8 \cdot n-9}{2 \cdot 3 \cdot 4 \cdot 5 r^{n-11}} x^{n-10}$$

\&c. $= \pm c$, according as $\frac{n+1}{2}$ is odd or even, the Se-

ries continued to $\frac{n+1}{2}$ Terms: that is, $+c$ when odd, and $-c$ when even.

$$\text{Or thus; } \sqrt{4rr - xx} : X \text{ into } \frac{x^{n-1}}{r^{n-1}} - \frac{n-2 \cdot x^{n-3}}{r^{n-3}} + \frac{n-3 \cdot n-4}{1 \cdot 2 r^{n-5}} x^{n-5} - \frac{n-4 \cdot n-5 \cdot n-6}{1 \cdot 2 \cdot 3 r^{n-7}} x^{n-7} + \frac{n-5 \cdot n-6 \cdot n-7 \cdot n-8}{1 \cdot 2 \cdot 3 \cdot 4 r^{n-9}} x^{n-9} - \text{ \&c.} = \pm c, \text{ according}$$

as $\frac{1}{2}n$ is odd or even, and the Series continu'd to $\frac{n}{2}$ Terms.

For the former Series is the Root of $x^{2n} - 2nx^{2n-2} + 2n \times \frac{2n-3}{2} x^{2n-4} \text{ \&c. the Square of the Cord, found}$

FIG. above by the first Method. And the latter Series is found by dividing the Series $x^{2n} - 2nx^{2n-2} \&c.$ by $xx+4$ (or rather by $xx-4$), and extracting the Root of the Quotient.

Cor. 2. If v be the versed Sine of an Arch, V the versed sine of n times the Arch, then will $\frac{2v^n}{2r^n} -$

$$\frac{2nr}{2v} A - \frac{2n-3 \cdot r}{2 \cdot 2v} B - \frac{2n-4 \cdot 2n-5 \cdot r}{3 \cdot 2n-3 \cdot 2v} C -$$

$$\frac{2n-6 \cdot 2n-7 \cdot r}{4 \cdot 2n-4 \cdot 2v} D - \frac{2n-8 \cdot 2n-9 \cdot r}{5 \cdot 2n-5 \cdot 2v} E \&c. = \pm V,$$

according as n is odd or even, the Series continued to n Terms.

Or $V = n^2 v - \frac{n^2 - 1 \cdot 2v}{2 \cdot 2 \cdot 3r} A - \frac{2 \cdot n^2 - 2^2 \cdot 2v}{3 \cdot 4 \cdot 5r} B -$

$$\frac{3 \cdot n^2 - 3^2 \cdot 2v}{4 \cdot 6 \cdot 7r} C - \frac{4 \cdot n^2 - 4^2 \cdot 2v}{5 \cdot 8 \cdot 9r} D - \&c.$$

This appears by substituting $2rv$ for xx , and $2rV$ for bb , to which they are equal, by Cor. 8. Prop. I.

PROP. XXX.

The Cord of the Supplement of an Arch being given; to find the Cord of the Supplement of n times the Arch.

Let Radius $= r = 1$. $x =$ Cord of the Supplement, $A =$ Arch; then by Cor. Prop. XXIII. $1 : x ::$ Cord. Sup. $nA ::$ Cord. Sup. $n-1 \cdot A +$ Cord. Sup. $n+1 \cdot A$. whence Cord. Sup. $n+1 \cdot A = x \times$ Cord. Sup. $nA -$ Cord. Sup. $n-1 \cdot A$. that is, if the Cord of the Sup. of any Arch, in a Series thereof, be multiply'd by x , and the preceding Cord of the Sup. subtracted, you will have the next succeeding one. Hence

Cord

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FIG.

Cord Sup. $0A=2$

$$1A=x$$

$$2A=x^2-2$$

$$3A=x^3-3x$$

$$4A=x^4-4x^2+2$$

$$5A=x^5-5x^3+5x$$

$$6A=x^6-6x^4+9x^2-2$$

$\&c.$

and in general $nA=x^n-ax^{n-2}+bx^{n-4}-cx^{n-6}\&c.$

then $n+1.A=x^{n+1}-ax^{n-1}+bx^{n-3}-cx^{n-5}\&c.$

$$= \left\{ \begin{array}{l} x^{n+1}-ax^{n-1}+bx^{n-3}-cx^{n-5} \\ -1x^{n-1}+ax^{n-3}-bx^{n-5} \end{array} \right\} \&c. \text{ by}$$

struction. Hence $a=a+1$, and $a=1=n$, whence $a=n$.

Also $b=b+a$, and $b=a=n$, whence $b=\frac{n-1}{2}$,

for when $n=4$, $b=2$. therefore $b=\frac{n-1}{2}+n-1=$

$$\frac{n-1}{2}+n-1=\frac{n-1}{2}.$$

Again, $c=c+b$, and $c=b=\frac{n-1}{2}+n$, and $c=$

$$\frac{n-1}{2}+n=\frac{n-1}{2}.$$

Again, $d=c$, $=\frac{n-1}{2}+n-1$, and $d=\frac{n-1}{2}+n-1$

$$+\frac{n-1}{2}+n-1=\frac{n-1}{2}.$$

In like manner $e=$

$$\frac{n-1}{2}+n-1=\frac{n-1}{2}.\&c. \text{ whence}$$

The

FIG. The Cord of the Supplement of the n^{th} Arch is $= x^{\text{th}}$

$$\begin{aligned}
 & -nx^{n-2} + \frac{n \cdot n-3}{1 \cdot 2} x^{n-4} - \frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} x^{n-6} + \text{Ec.} = \frac{x^n}{r^{n-1}} \\
 & - \frac{n \cdot r r}{x x} A - \frac{n-3 \cdot r r}{2 x x} B - \frac{n-4 \cdot n-5 \cdot r r}{3 \cdot n-3 \cdot x x} C - \\
 & \frac{n-6 \cdot n-7 \cdot r r}{4 \cdot n-4 \cdot x x} D - \frac{n-8 \cdot n-9 \cdot r r}{5 \cdot n-5 \cdot x x} E - \text{Ec. continued} \\
 & \text{to } \frac{n+1}{2} \text{ terms when } n \text{ is odd, or to } \frac{n+2}{2} \text{ terms when} \\
 & \text{even.}
 \end{aligned}$$

Cor. Hence if Radius $= r$. $x =$ Cofine of an Arch, then the Cofine of the n^{th} Arch $= \frac{2x^n}{2r^{n-1}} - \frac{nrr}{4 \cdot x x} A - \frac{n-3 \cdot r r}{4 \cdot 2 x x} B - \frac{n-4 \cdot n-5 \cdot r r}{4 \cdot 3 \cdot n-3 \cdot x x} C - \frac{n-6 \cdot n-7 \cdot r r}{4 \cdot 4 \cdot n-4 \cdot x x} D - \text{Ec.}$ continued to $\frac{n+1}{2}$ terms when n is odd, or to $\frac{1}{2}n+1$ terms when even.

This follows from the foregoing Series putting $\frac{1}{2}r$ for r .

P R O P. XXXI.

Given the Cord of an Arch, and the Cord of the Supplement of twice the Arch; to find the Cord of $2m+1$ times the Arch.

Let Radius $= r = 1$. Arch $= A$. Cord of $A = y$, Cord Sup. of $2A = z$. Then by Cor. Pr. XXIII. $1 : z ::$ Cord of $2m+1 \cdot A : \text{Cord of } 2m-1 \cdot A + \text{Cord of } 2m+3 \cdot A$, therefore Cord of $2m+3 \cdot A = z \times \text{Cord of } 2m+1 \cdot A - \text{Cord of } 2m-1 \cdot A$, that is in a Series of these Cords, if any one be multiply'd by z , and the foregoing one subtracted, you will have the next following one. Whence

Cord

Cord $A=y$

$$3A=z+1:xy$$

$$5A=z^2+z-1:xy$$

$$7A=z^3+z^2-2z-1:xy$$

$$9A=z^4+z^3-3z^2-2z+1:xy$$

$$11A=z^5+z^4-4z^3-3z^2+3z+1:xy$$

$$13A=z^6+z^5-5z^4-4z^3+6z^2+3z-1:xy$$

$$15A=z^7+z^6-6z^5-5z^4+10z^3+6z^2-4z-1:xy$$

and in general Cord $2m+1.A = z^m + z^{m-1} - az^{m-2} - bz^{m-3} + cz^{m-4} + dz^{m-5} - ez^{m-6} - fz^{m-7} \&c.$ then by Construction,

$$\left\{ \begin{array}{l} z^{m+1} + z^m - az^{m-1} - bz^{m-2} + cz^{m-3} + dz^{m-4} - ez^{m-5} \&c. \\ \quad \quad \quad - z^{m-1} - z^{m-2} + az^{m-3} + bz^{m-4} - cz^{m-5} \end{array} \right.$$

$$= z^{m+1} + z^m - az^{m-1} - bz^{m-2} + cz^{m-3} + dz^{m-4} - ez^{m-5}$$

$\&c. =$ Cord of $2m+3.A$. whence $a=a+1$, and

$a-a$ or $a=1=m$, whence $a=m$, for when $m=2$, $a=1$.

Again, $b-b$ or $b=1=m$, and $b=m$, for when $m=$

3, $b=1$.

Also $c=c+a$, and $c-c$ or $c=m$, whence $c=\frac{-3-2}{2}$.

Likewise $d=d$ or $d=b=m$, and $d=\frac{-4-3}{2}$.

Also $e=c=\frac{-4-3}{2}$, and $e=\frac{-5-4-3}{2.3}$, likewise $f=d$,

and $f=\frac{-6-5-4}{2.3}$, $g=\frac{-7-6-5-4}{2.3.4}$, $h=\frac{-8-7-6-5}{2.3.4}$, $i=$

$\frac{-9-8-7-6-5}{2.3.4.5}$, $k=\frac{-10-9-8-7-6}{2.3.4.5} \&c.$ whence the

Cord of the $2m+1$. Arch $= y \times$ into $z^m + z^{m-1} -$

FIG. $\frac{m-1}{1} z^{m-2} - \frac{m-2}{1} z^{m-3} + \frac{m-2 \cdot m-3}{2} z^{m-4} +$
 $\frac{m-3 \cdot m-4}{2} z^{m-5} - \frac{m-3 \cdot m-4 \cdot m-5}{2 \cdot 3} z^{m-6} -$
 $\frac{m-4 \cdot m-5 \cdot m-6}{2 \cdot 3} z^{m-7} + \frac{m-4 \cdot \dots \cdot m-7}{2 \cdot 3 \cdot 4} z^{m-8} +$
 $\frac{m-5 \cdot \dots \cdot m-8}{2 \cdot 3 \cdot 4} z^{m-9} - \frac{m-5 \cdot \dots \cdot m-9}{2 \cdot 3 \cdot 4 \cdot 5} z^{m-10} -$
 $\frac{m-6 \cdot \dots \cdot m-10}{2 \cdot 3 \cdot 4 \cdot 5} z^{m-11} \text{ \&c. continued to } m+1$
 Terms.

PROP. XXXII.

Given the Cord of the Supplement of an Arch, and the Cord of the Supplement of twice the Arch; to find the Cord of the Supplement of $2m+1$ times the Arch.

Let Radius = 1, Arch = A, Cord Sup. A = x, Cord Sup. $2A = z$, then as in the last Prop. Cord. Sup. $2m+3 \cdot A = z \times$ Cord. Sup. $2m+1 \cdot A -$ Cord. Sup. $2m-1 \cdot A$. that is, if, in a Series, any one be multiply'd by z , and the foregoing subtracted, you will have the next following one. Hence Cord. Sup.

of $A = x$

of $3A = z - 1 : x x$

of $5A = z^2 - z - 1 \times x x$

$7A = z^3 - z^2 - 2z + 1 \times x x$

$9A = z^4 - z^3 - z^2 + 2z + 1 \times x x$

$11A = z^5 - z^4 - 4z^3 + 3z^2 + 3z - 1 \times x x$

\&c.

And in general, Cord. Sup. $2m+1 \cdot A = z^m - z^{m-1} -$
 $az^{m-2} + bz^{m-3} + cz^{m-4} - dz^{m-5} - ez^{m-6} \text{ \&c. there-}$
 fore by Construction

$$\left. \begin{array}{l} z^{m+1} - z^m - az^{m-1} + bz^{m-2} + cz^{m-3} \\ - 1z^{m-1} + 1z^{m-2} + az^{m-3} \\ - 1 \end{array} \right\} \text{ \&c. } =$$

$$z^{m+1} - z^m - \underset{1}{az^{m-1}} + \underset{1}{bz^{m-2}} + \underset{1}{cz^{m-3}} = \text{Cord. Sup.}$$

$2m+3$

$2m+3$. A. hence $a=a+1$, $b=b+1$, $c=c+1$, $d=d+1$ FIG.

$\&c$. all as in the last Prop. and therefore the Coefficients a , b , c , $\&c$. will be the very same, whence the

$$\begin{aligned} \text{Cord. Sup. } 2m+1. A = x \times \text{ into } & z^m - z^{m-1} - \frac{m-1}{1} z^{m-2} \\ & + \frac{m-2}{1} z^{m-3} + \frac{m-2 \cdot m-3}{2} z^{m-4} - \frac{m-3 \cdot m-4}{2} z^{m-5} \\ & - \frac{m-3 \cdot m-4 \cdot m-5}{2 \cdot 3} z^{m-6} + \frac{m-4 \cdot m-5 \cdot m-6}{2 \cdot 3} z^{m-7} + \\ & \frac{m-4 \cdot \dots \cdot m-7}{2 \cdot 3 \cdot 4} z^{m-8} \&c. \end{aligned}$$

Cor. If Rad. = 1, $z = \text{Cord. Sup. of } \frac{1}{2m+1}$ Part of the Circumference, then $0 = z^m - z^{m-1} - \frac{m-1}{1} z^{m-2} + \frac{m-2}{1} z^{m-3} + \frac{m-2 \cdot m-3}{2} z^{m-4} \&c$.

For if $2m+1 \times A = \text{Semi-circumference}$, then the Cord of the Sup. and the general Series is = 0, and may be divided by x . And in that Series $z = \text{Cord. Sup. } 2A$, or of the $\frac{2m+1}{2}$ Part of the Circumference.

P R O P. XXXIII.

The Tangent of an Arch A being given; to find the Sine and Cosine of n times the Arch.

Let $t = \text{Tan. } A$, then by Prop. XXIV. $S.nA = \frac{nx^{n-1}y}{r^{n-1}} - \frac{n-1 \cdot n-2 \cdot yy}{2 \cdot 3 \cdot xx} A \&c$. but $\frac{y}{x} = \frac{t}{r}$, and $x = \frac{rr}{\sqrt{rr+tt}}$, therefore $\frac{nx^{n-1}y}{r^{n-1}}$ or $\frac{nx^n r y}{r^n x} = \frac{nx^n t}{r^n} = \frac{nr^n t}{(rr+tt)^{\frac{n}{2}}}$, whence

$$S.nA = \frac{nr^n t}{(rr+tt)^{\frac{n}{2}}} - \frac{n-1 \cdot n-2 \cdot tt}{2 \cdot 3 \cdot rr} A \&c.$$

Like-

FIG.

Likewise by Prop. XXVI. Cof. $nA = \frac{x^n}{r^n - 1} -$

$$\frac{n \cdot n - 1 \cdot yy}{2 \cdot x \cdot x} A \text{ \&c. that is } = \frac{r^{n+1}}{rr+tt} - \frac{n \cdot n - 1 \cdot tt}{2 \cdot rr} A \text{ \&c.}$$

whence

$$S.nA = \frac{nr^n t}{rr+tt} - \frac{n-1 \cdot n-2 \cdot tt}{2 \cdot 3 \cdot rr} A - \frac{n-3 \cdot n-4 \cdot tt}{4 \cdot 5 \cdot rr} B$$

$$- \frac{n-5 \cdot n-6 \cdot tt}{6 \cdot 7 \cdot rr} C - \frac{n-7 \cdot n-8 \cdot tt}{8 \cdot 9 \cdot rr} D - \text{\&c. and}$$

$$\text{Cof. } nA = \frac{r^{n+1}}{rr+tt} - \frac{n \cdot n - 1 \cdot tt}{2 \cdot rr} A - \frac{n-2 \cdot n-3 \cdot tt}{3 \cdot 4 \cdot rr} B -$$

$$\frac{n-4 \cdot n-5 \cdot tt}{5 \cdot 6 \cdot rr} C - \frac{n-6 \cdot n-7 \cdot tt}{7 \cdot 8 \cdot rr} D \text{ \&c.}$$

Cor. 1. Hence since $t = \frac{ry}{x}$, therefore $\text{Tan. } nA =$

$$nr^n t - \frac{n \cdot n - 1 \cdot n - 2}{2 \cdot 3} r^{n-2} t^3 + \frac{n \cdot \dots \cdot n - 4}{2 \cdot 3 \cdot 4 \cdot 5} r^{n-4} t^5 \text{ \&c.}$$

$$r^n - \frac{n \cdot n - 1}{1 \cdot 2} r^{n-2} t^2 + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{2 \cdot 3 \cdot 4} r^{n-4} t^4 \text{ \&c.}$$

Cor. 2. If $f = \text{Secant of } A$, and $t = \text{Tan. } A$, then

$$\text{Secant } nA = \frac{f^n}{r^{n-1} - \frac{n \cdot n - 1}{2} r^{n-3} t^2 + \frac{n \cdot \dots \cdot n - 3}{1 \cdot \dots \cdot 4} r^{n-5} t^4 - \text{\&c.}}$$

for $\sqrt{rr+tt} = f$, and $\frac{\text{Rad}^2}{\text{Cofine}} = \text{Secant.}$

S C H O L I U M.

Tho' the principal Use of the foregoing Propositions is when n is an integer Number, in which Case the Series will always terminate. Yet these Series will be equally true when n is any fractional Number, if they do but converge. For the Investigation of any of these Propositions does not suppose that n is always an Integer, it only supposes that n is 1.

S E C T.

S E C T. V.

Angular Sections, the Inscription of Polygons, the Properties of the Cords or Subtenses of Arches.

P R O P. XXXIV.

If any Arch of a Circle as AR be given, and there be taken the Arch $AD = \frac{1}{n}$ the Arch AR; n being any integer Number; and if the whole Circumference of the Circle be divided into n equal Parts at the Points D, E, F, G, H, &c. beginning at D: Then I say any of the Arches AD, AE, ADF, AEG, AEH, &c. repeated n times, from A, will always end at R. 7.

Call the whole Circumference C, then by Construction, $n \times AD = AR$, also $n \times DE = C$, and $n \times DEF = 2C$, and $n \times DEG = 3C$, $n \times DFH = 4C$, &c. therefore

$$\begin{aligned} n \times AE \text{ or } n \times AD + n \times DE &= AR + C \\ \text{and } n \times AF \text{ or } n \times AD + n \times DF &= AR + 2C \\ \text{and } n \times AEG \text{ or } n \times AD + n \times DEG &= AR + 3C \\ n \times AFH \text{ or } n \times AD + n \times EFH &= AR + 4C, \text{ \&c.} \end{aligned}$$

Therefore $n \times AE$, $n \times AF$, $n \times AEG$, &c. $= 1, 2, 3$, &c. entire Circumferences + the Arch AR. And since any Number of Circumferences will end at the Point A where they began; consequently by the Addition of the Arch AR, they will necessarily all end at R.

Cor. If the Cords AR, BR, AD, AE, AF, &c. are drawn; then the Cord AR = Cord of n times AD, or of n times AE, or of n times AEF, or of n times AEG, &c. and BR is the Cord of the Supplement of each of these.

P R O P.

FIG.

PROP. XXXV.

To divide an Arch AR into any Number of equal Parts as n .

Let the Radius $= 1$, $b =$ Cord of the Sup. of AR, or the Cord of BR, $x =$ BD the Cord Sup. first Part AD. Then by Prop. XXX. $x^n - nx^{n-2} + \frac{n \cdot n - 3}{2} x^{n-4}$

$\&c. = \pm b$, according as the Arch AR is lesser or greater than a Semi-circle. Therefore x the greatest Root of this Equation, being extracted, will give BD the Cord of the Supplement of the Arch required.

EXAMPLE.

If the Arch AR be less than a Semi-circle, and is to be divided into 7 equal Parts; then $n=7$, and you will have $x^7 - 7x^5 + 14x^3 - 7x = b$, whose greatest Root x is BD, the Cord of the Supplement of AD required.

Otherwise.

Let Rad. $= 1$, Cord AR $= b$, Cord AD $= x$, then by Prop. XXIX. you will have $x^{2n} - 2nx^{2n-2} + 2n \times \frac{2n-3}{2} x^{2n-4}$, $\&c.$ to n terms $= \pm b b$, according as n

is odd or even. Then extract the least Root xx of this Equation, and then its Square Root x will be $=$ AD, the Cord of the first of the equal Parts. Or thus,

$$\text{put } n n x x - \frac{n n \cdot n n - 1}{2 \cdot 1 \cdot 2 \cdot 3} x^4 + \frac{n n \cdot n n - 1 \cdot n n - 4}{3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^6 \&c. = b b, \&c.$$

EXAMPLE.

If AR was to be divided into 5 equal Parts, then $n=5$, and $x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 = b b$, whose least Root xx is the Square of AD, and $x =$ AD, the Cord of $\frac{1}{n}$ Part of the Arch AR.

Cor.

Cor. If you continue to extract all the Roots x , you will get the Values of all the Cords BE, BF, BG, &c. by the former Method; or of the Cords AE, AF, AG, &c. by the latter Method; as appears by Cor. Prop. XXXII. FIG. 7.

S C H O L I U M.

If n is not a prime Number, but compounded of two or more Primes; it will be easier to divide the Arch into a Number of Parts denoted by one of these compounding Numbers, and then the first of these Parts into as many Parts as is denoted by another of them, and so on to the last.

P R O P. XXXVI.

To inscribe any regular Polygon in a Circle.

Let the Number of its Sides be $=2m+1$, an odd Number. Radius $=1$, z = the unknown Cord BD, or the Cord of the Supplement of one Part AD. 8.

Then by Cor. Pr. XXXII. $z^m - z^{m-1} - \frac{m-1}{1} z^{m-2} +$

$\frac{m-2}{1} z^{m-3}$, &c. $=0$, therefore the greatest Root z

will give BD, and $\sqrt{4-zz}$ = Side of the Polygon AD, required.

E X A M P L E.

In a Heptagon, $2m+1=7$, or $m=3$, then by the former Equation $z^3 - z^2 - 2z + 1 = 0$, and the greatest Root z gives the Value of BE, and $\sqrt{4-zz}$ = AE the Side of the Heptagon. 9.

Otherwise.

Let n = Number of Sides, Rad. $=1$. x = Cord of the Sup. of the n^{th} Part of the Circle. Then since the Cord subtending the whole Circle is 0, therefore

F

by

FIG. by Prop. XXVIII. $yx^{n-1} - \frac{n-2}{1}yx^{n-3}, \&c. = 0$, therefore dividing by y , you will have $x^{n-1} - \frac{n-2}{1}x^{n-3} + \frac{n-3 \cdot n-4}{1 \cdot 2}x^{n-5}, \&c. = 0$, extract the greatest Root x of this Equation, then $\sqrt{4-xx} = \text{Side of the Polygon.}$

E X A M P L E.

In a Pentagon $n=5$, then $x^4 - 3x^2 + 1 = 0$, and $x^2 = \text{the greatest Root, and } \sqrt{4-xx} = \text{Side of the Pentagon.}$

S C H O L I U M.

When the Number of Sides is not prime; it will be easier to divide the Circumference into a Number of Parts denoted by one of the Primes that compose it; and then one of these Parts into as many Parts as is denoted by another of the compounding prime Numbers, and so on till all the Primes which compose the given Number be exhausted.

P R O P. XXXVII.

10. *If the Arch DAF of a Circle, whose Diameter is AE, be divided into an even Number of equal Parts at C, B, A, H, G, &c. and there be drawn the lines BH, CG, DF, &c. perpendicular to the Diameter AE; and likewise the Cords AB, BE. I say, the Sum of the Lines, BH, CG, &c. + half the last, that is BH + CG + DO = $\frac{AO \times EB}{AB}$.*

For draw the Lines CH, DG, &c. then all the Triangles BAK, KHL, LCM, MGN, NDO, &c. are similar. Therefore $BK : KA :: HK : KL :: CM : ML :: GM : MN :: DO : ON$, &c. therefore $BK : AK :: BK + KH + CM + GM + DO : AK + KL + LM + MN + NO :: BH + CG + DO : AO$.

: O A. But $EB : AB :: BK : AK :: BH + CG + DO : AO$, therefore $AB \times BH + CG + DO = AO \times EB$. 10.

Cor. 1. If the Circle A D G be divided into an even Number of equal Parts at A, B, C, D, E, F, &c. and all the Lines B H, C G, D F, &c. be drawn, and also A B, B E. Then the Sum of all the Lines B H + C G + D F, &c. $= \frac{AE \times EB}{AB}$.

For here the base of the Segment is o, and the Points D, O, F, E coincide.

Cor. 2. If a Semicircle A B E be divided into any Number of equal Parts as n , at the Points B, C, D, &c. and the Sines B K, C M, D O, &c. be drawn; then I say, the Sum of all the Sines, $BK + CM + DO$, &c. $= \frac{AE \times EK}{2BK}$.

For by similar Triangles $\frac{EB}{AB} = \frac{EK}{BK}$; and taking half of the Quantities in Cor. 1. $BK + CM + DO = \frac{AE \times EB}{2AB} = \frac{AE \times EK}{2BK}$. Hence

Cor. 3. Let Radius $= r$, $\frac{180}{n} = A$, n being any Integer. $s = S.A.$ $c = \text{Cof. } A$. then the Sum of all the Sines of $A, 2A, 3A, 4A$, &c. in the Semicircle, is $= \frac{r+c}{s} r$.

Cor. 4. Hence the Area of the Curve, made by erecting all the Sines perpendicular to a Line equal to the Semi-circumference, or as it is commonly called, the Area of the Figure of Sines, is $= 2rr$.

For if n be infinite and A be $= 1 = s$, then $c = r$, and the Sum of all the Sines or Area of the Curve $= \frac{2rr}{s} = 2rr$.

FIG.

10.

P R O P. XXXVIII.

If the Circumference of the Circle ADG be divided into an even Number of equal Parts, at the Points A, B, C, D, &c. and the parallel Chords AB, CH, DG, &c. be drawn; I say, the Sum of all the Cords $AB + CH + DG + EF$, &c. $= \frac{AE^2}{AB}$.

For by similar Triangles $AE : AB :: AB : AK :: HL : KL :: CL : ML :: GN : MN :: DN : NO :: FE : EO$, and by Composition $AE : AB :: AB + CH + DG + FE : AE$.

Cor. 1. If a Circle is divided into an even Number of equal Parts, and A be one of the Parts. Then the Sum of all the Cords of $2A, 4A, 6A, 8A$, &c. is to the Sum of all the Cords of $A, 3A, 5A, 7A$, &c. :: as EB : to EA .

For BH, CG, DF , &c. are the Cords of $2A, 4A, 6A$. And BA, CH, DG , &c. are the Cords of $1A, 3A, 5A$, &c. And the first Sum, is to the last as $\frac{AE \times EB}{AB}$ to $\frac{AE^2}{AB}$ or as EB : to EA .

Cor. 2. If a Semicircle is divided into any even Number of equal Parts, as n ; and if A be one of these Parts, and $s = S.A$, Radius $= r$; then the Sum of the Sines of $A, 3A, 5A, 7A$, &c. $= \frac{rr}{s}$.

This follows from this Prop. taking half the Cord for the Sine of half the Arch.

Cor. 3. If a Semicircle is divided into an even Number of equal Parts, and A be one of these Parts, and Rad. $= r$, $S.A = s$, Cos. $A = c$, then the Sum of all the even Sines in the Semicircle, that is the Sum of the Sines of $2A, 4A, 6A, 8A$, &c. $= \frac{cr}{s}$.

For

For by Cor. 3. Pr. 37. } $A, 2A, 3A, 4A \text{ \&c. } = \frac{r+c}{s} r$ FIG.

Sum of the Sines of } $A, 3A, 5A \text{ \&c. } = \frac{rr}{s}$

And by the last Cor. } $A, 3A, 5A \text{ \&c. } = \frac{rr}{s}$

Sum of the Sines of } $A, 3A, 5A \text{ \&c. } = \frac{rr}{s}$

There Sum } $2A, 4A, 6A \text{ \&c. } = \frac{cr}{s}$

P R O P. XXXIX.

If the Circumference of a Circle, whose Diameter is AB and Center C, be divided into any odd Number of equal Parts, as n , at the Points D, E, F, G, H, I, K; and if, from any Point A, there be drawn the Cords AD, AE, AF, AG, AH, &c. and if there be taken the Arch $AR = n \times$ Arch AD, and the Cord AR drawn: I say the Product of all the Cords $AD \times AE \times AF \times AG \times AH \times AI \times AK = AR \times CB^{n-1}$.

7.

1. Let the Line QAP be supposed to revolve about the Point A, the common Origin of all the Arches; then will the Part AP pass thro' the Circumference of the Circle AEBHA. And if the Line still continues to revolve about A, and the other End AQ will also pass thro' all the Points of the Circumference AEBHA. And if the Line still revolves forward, then the Part AP passes again thro' the Circle AEBHA, &c. and so on alternately. Now if from A to any Point of Intersection as R, the Cord AR be drawn, this Cord, AR will be affirmative in going the first time about the Circle; but negative, the second time about; because then AR will lie from A towards Q, in the Revolving line, which is the contrary way to what it laid before; likewise in going the third time about, AR will be affirmative, and the fourth time negative, and so on alternately. Therefore in general, let $C =$ Circumference, $A =$ any Arch less than C ; then the Cords of $A, 2C+A, 4C+A, \text{ \&c. }$ or of any even Number of C 's $+A$, will be affirmative. And the Cords of

F 3

 $C+A,$

FIG. $C+A$, $3C+A$, $5C+A$, &c. or of any odd Number of C 's $+A$, will be negative.

7. 2. Let $AR=c$, x =Cord of the single Arch. Now since n times the Arch AD , AE , AF &c. all end at R (by Prop. XXXIV.) and therefore have the same Cord AR , consequently by Cor. 1. Pr. XXIX. we shall have

Case 1. $x^n - nx^{n-2} \&c. = +c$, the Cords of A , $2C+A$, $4C+A$ &c.

or the Cords of $nxAD$, $nxAF$, $nxAH$ &c.

Case 2. $x^n - nx^{n-2} \&c. = -c$, the Cords of $C+A$, $3C+A$, $5C+A$ &c.

or the Cords of $nxAE$, $nxAG$, $nxAEH$ &c.

And in Case 1st the Root AD , being affirmative, the others AF , AH &c. are affirmative for the same Reason.

And in Case 2^d the Root AE is affirmative, and likewise the Roots AG , AI . But it is shewn in Algebra, that if the Signs of all the even Powers in an Equation be changed, the affirmative Roots are changed into negatives, and the contrary. And in Case 2^d all the even Powers of x are wanting, except c , therefore in Case 2^d changing $-c$ into $+c$, we have $x^n - nx^{n-2} \&c. = +c$, in which Equation, AE , AG , AI &c. are negative Roots. But this Equation is now the same as Case 1. therefore in general, in the Equation $x^n - nx^{n-2} \&c. = c$, the affirmative Roots are AD , AF , AH &c. and the negative ones AE , AG , AI &c..

3. It is shewn in Algebra that in any Equation, as $x^n - nx^{n-2} \&c. = c$, or rather $x^n - nr^2x^{n-2} \&c. = cr^{n-1}$, the absolute Number is equal to the Product of all the Roots; therefore $AD \times AE \times AF \times AG \&c. = cr^{n-1}$.

Cor. 1. The Sum of all the odd Cords is equal to the Sum of all the even ones, $AD+AF+AH+AK=AE+AG+AI$.

For it is shewn in Algebra, that the Coefficient of the second Term is = Sum of all the Roots; and here the

the second Term is wanting, and therefore $AD - AE + AF - AG \&c. = 0$. FIG.

Cor. 2. If the Semi-circumference be divided into any odd Number of equal Parts, as n ; whereof the two first are BF , the two next EF , the two next $ED \&c.$ and if the Cords AF , AE , $AD \&c.$ be drawn: Then the Product of all these Cords, $AF \times AE \times AD = CB^{\frac{n-1}{2}}$. 11.

For if $AD = AK$, then R and G fall in B , and $AE = AI$, $AF = AH \&c.$ therefore by this Prop. $AD^2 \times AE^2 \times AF^2 \times AB = AB \times r^{n-1}$. 7.

Cor. 3. The same Things supposed as in the last Cor. I say, $AF - AE + AD \&c. = CB$.

For by Cor. 1. $2AD + 2AF = 2AE + AB$.

Cor. 4. If Arch $BR = n \times BG$, and the Cord AR drawn, and the rest be done as is supposed in the Prop. then the Product of all the Cords $AD \times AE \times AF \&c. = AR \times CB^{n-1}$, and $AD + AF + AH + AK = AE + AG + AI$.

For it will appear with a little Consideration, that $AD + BG = \text{half of } DE = \frac{C}{2n}$, or $n \times AD + n \times BG = \frac{1}{2}C$. But $n \times BG = BR$, therefore $n \times AD = \frac{1}{2}C - BR = AR$, therefore this Cor. comes to the Case of this Proposition.

Cor. 5. If a Quadrant be divided into any odd Number of equal Parts, as n ; and A be one of these Parts, Radius $= r$. then the Product of all the Sines, of $1A$, $3A$, $5A$, $7A \&c.$ in the Quadrant =

$$2 \times \frac{r}{2} \left[\frac{n+1}{2} \right]$$

This follows from Cor. 2. taking half the Cord for the Sine of half the Arch, and half Radius for CB , and multiplying both Sides by r .

P R O P. XL.

9. If the Circumference of a Circle, whole Center is C, be divided into any odd Number of equal Parts as n , at the Points A, E, F, G, H, I, K, and from any one as A, to all the rest there be drawn the Cords A E, A F, A G &c. I say the Product of all the Cords, $A E \times A F \times A G \times A H \times A I \times A K = n \times A C^{n-1}$.

7. For let the Arches A D, A R be infinitely small, and they will coincide with their Cords; then the Cord A D, is to the Cord A R, as 1 to n , and with this Ratio the Cords A D, A R vanish when A and D coincide, and therefore by Prop. XXXVII. $1 \times A E \times A F \times A G \text{ \&c.} = n \times C B^{n-1}$.

Otherwise.

If A D be $=0$, then n times the Arch A E, A F, A G &c. always ends at A, and therefore c the Cord of these multiple Arches is $=0$, therefore if $s =$ Cord of the single Arch, we shall have (by Cor. 2. Prop.

XXV.) $ns - \frac{n \cdot nn - 1}{24r^2} s^3 \text{ \&c.} = 0$, wherein one Root s is $=0$, and the Equation being divided, we have $n - \frac{n^3 - n}{24r^2} s^2 \text{ \&c.} = 0$, wherein the absolute Number is n or rather nr^{n-1} , which therefore is equal to the Product of all the Roots, whose Number is now $n-1$.

9. Cor. 1. If the Semi-circumference A E B be divided into any odd Number of equal Parts, as n ; whereof the two first is A E, the two next E F, the two next F G &c. And the Cords A E, A F, A G &c. be drawn: I say the Product of all the Cords $A E \times A F$

$$\times A G \text{ \&c.} = C B^{\frac{n-1}{2}} \times \sqrt{n}.$$

For $A E = A K$, $A F = A I$ &c. therefore $A E^2 \times A F^2 \times A G^2 = n \times A C^{n-1}$.

Cor.

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Cor. 2. If the Quadrant of a Circle be divided into any odd Number of equal Parts as n , and A be one of them; then the Product of all the Cosines of A, FIG. 9.

$$3A, 5A, 7A \text{ \&c.} = \frac{r^{n-1}}{2} \times \sqrt{n}.$$

This follows from the last Cor. because $\frac{1}{2}AG, \frac{1}{2}AF \text{ \&c.}$ are the Cosines of $\frac{1}{2}BG, \frac{1}{2}BF \text{ \&c.}$ or of $A, 3A \text{ \&c.}$

Cor. 3. Hence also, the same things supposed as in this Prop. the sum of the Squares of the Cords $AE^2 + AF^2 + AG^2 + AH^2 + AI^2 + AK^2 = 2n \times CB^2$.

For in the Equation $ns - \frac{n^3 - n}{24rr} s^3 \text{ \&c.} = 0$, or rather by Cor. 1. Prop. XXIX. $x^n - nx^{n-2} \text{ \&c.} = 0$, one Root $x=0$, which being divided, $x^{n-1} - nx^{n-3} \text{ \&c.} = 0$, since here the Index decreases by 2, therefore xx is the Root of the Equation, and n or nr^2 the Coefficient of the second Term is the Sum of the Roots, that is of the Squares of AE, AF, AG ; which doubled gives the Sum of all the Squares.

P R O P. XLI.

If the Circumference of a Circle, whose Center is C, and Diameter AB, be divided into any even Number of equal Parts, at the Points D, E, F, G, H, I, and from any Point A, there be drawn the Cords AD, AE, AF \&c. And the Arch AR be taken $= n \times$ Arch AD, and the Cord AR drawn; then the Product of all the Cords $AD \times AE \times AF \times AG \times AH \times AI = AR \times CB^{n-1}$. 121

Let Cord $AR=c$, and x = Cord of the single Arch, then by Prop. XXIX. we have $-x^{2n} + 2nx^{2n-2} \text{ \&c.} = cc$. But xx or the Roots of this Equation are $AD^2, AE^2, AF^2 \text{ \&c.}$ and therefore their Product $AD^2 \times AE^2 \text{ \&c.} = cc$ or rather ccr^{2n-2} , and extracting the Root, $AD \times AE \times \text{ \&c.} = cr^{n-1}$.

Cor.

FIG. Cor. 1. The Sum of the Squares of all the Cords $AD^2 + AE^2 + AF^2 + AG^2 + AH^2 + AI^2 = 2n \times CB^2$.

For the Coefficient of the second Term is the Sum of all the Roots $= 2n$ or $2nr^2$.

13. Cor. 2. If the Semicircle AEB be divided into an even Number of equal Parts, as n ; and the odd Cords AD, AE, AF drawn; I say their product $AD \times AE \times AF = CA^{\frac{n}{2}} \times \sqrt{2}$.

12. For in this Case, $AD = AI$, $AE = AH$, and $AF = AG$, and $AR = 2CB$, therefore $AD^2 \times AE^2 \times AF^2 = AR \times CB^{n-1} = 2CB^n$.

PROP. XLII.

8. If the Circumference of a Circle, whose Center is C, be divided into any even Number of equal Parts, at A, D, E &c. and from one of them as A, there be drawn to all the rest the Cords AD, AE, AB, AG, AH; I say the Product of all the Cords, $AD \times AE \times AB \times AG \times AH = n \times CB^{n-1}$.

For $n \times AD$, $n \times AE$ &c. all end at A, and their Cords $= 0$, therefore by Cor. 1. Prop. XXIX. $x^{n-1} - n-2 \cdot x^{n-3}$ &c. $\dots + \frac{1}{2}nx : \times \sqrt{4-xx} = 0$, and one Root $x=0$, and dividing the Equation by it and the Correspondent Cord of the Supplement $\sqrt{4-xx}$ (which here is AB), and we shall have $\frac{1}{2}x^{n-2} + n-2 x^{n-4} \dots + \frac{1}{2}n = 0$, where $\frac{1}{2}n$ or $\frac{1}{2}nr^{n-2}$ is the Rectangle of the remaining Roots AD, AE, AG, AH, which multiply by $2r$, and you have $AD \times AE \times AB$ &c. $= nr^{n-1}$.

This Prop. is also evident from the last Prop. by supposing AD, AR infinitely small, and to vanish in the Ratio of 1 to n .

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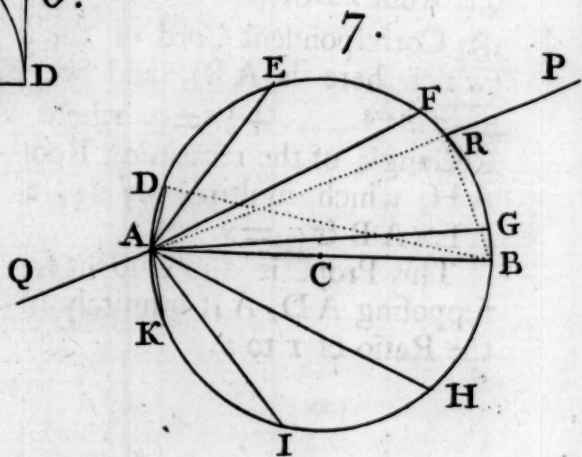
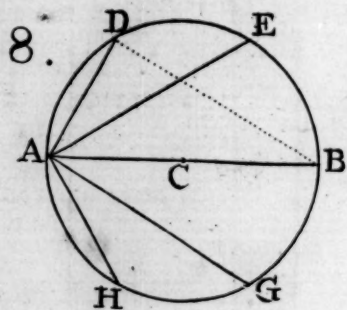
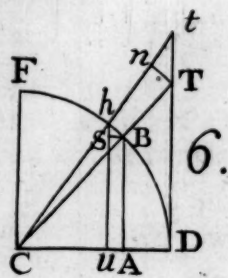
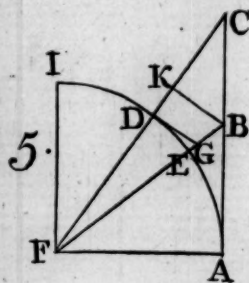
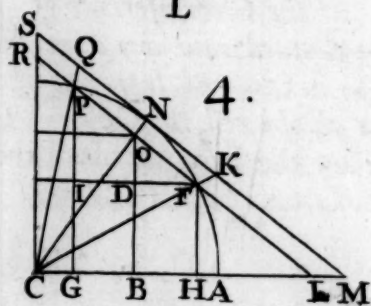
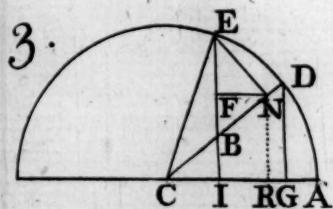
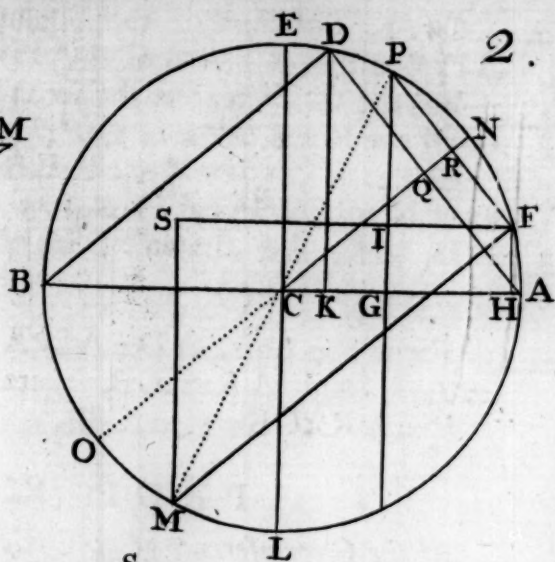
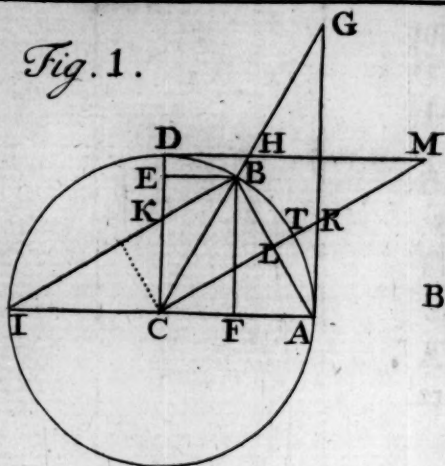
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PROP. XLIII.

If the Circumference of a Circle AFHA, whose Diameter is AB and Center C, be divided into any even Number of equal Parts as n , at the Points D, E, F, G, H, I; and from any Point B in the Circumference, there be drawn the Cords BD, BE, BF &c. and if the Arch AR be taken $= n \times$ Arch AD, and the Cord BR drawn; I say

14.

15.

The Product of the Squares of the odd Cords $BD^2 \times BF^2 \times BH^2 = BA + BR \times CB^{n-1}$, when $\frac{1}{2}n$ is an odd Number; and $= BA - BR \times CB^{n-1}$, when $\frac{1}{2}n$ is an even Number.

And the Product of the Squares of the even Cords $BE^2 \times BG^2 \times BI^2 = BA - BR \times CB^{n-1}$, when $\frac{1}{2}n$ is odd, and $= BA + BR \times CB^{n-1}$, when $\frac{1}{2}n$ is even.

For Let A be the common Origin of the Arches; and let the line PQ revolve about B, beginning at A, and going thro' the Semi-circumference AEB in which any Cord as BR being drawn, will be affirmative. But after the Point of Intersection passes thro' B, the other Part BQ passes thro' the whole Circumference, in which any Cord BR will be negative, then the Part PB passes thro' the next Circumference from B, and any Cord drawn in it will be affirmative; and so on alternately. Therefore if $C =$ Circumference, $A =$ Arch AR less than $\frac{1}{2}C$; then will the Cords of $A, 2C+A, 4C+A$ &c. be affirmative, and the Cords of $C+A, 3C+A, 5C+A$ &c. negative. Put $a =$ Cord of BR, then

CASE I.

In the Arches AD, AF, AH, whose Sup. Cords are BD, BF, BH; we have $n \times AD = A$, $n \times AF = 2C + A$, $n \times AH = 4C + A$ &c. wherefore the Cords of $n \times AD, n \times AF, n \times AH$ are affirmative.

Whence

FIG. Whence by Prop. XXX.

14. $x^n - nx^{n-2} \mathcal{E}c. - 2 = +a$, when $\frac{1}{2}n$ is odd,
 15. and $x^n - nx^{n-2} \mathcal{E}c. + 2 = +a$, when $\frac{1}{2}n$ is even,

that is $x^n - nx^{n-2} \mathcal{E}c. = a \pm 2$, according as $\frac{1}{2}n$ is odd or even. In which Equation the Squares of BD, BF, BH the odd Cords are the Roots; therefore their Product is $2 \pm a$, or rather $2r \pm a \times r^{n-1}$, according as $\frac{1}{2}n$ is odd or even.

CASE II.

In the Arches AE, AEG, AEI, whose Sup. Cords, are BE, BG, BI; we have $n \times AE = C + A$, $n \times AEG = 3C + A$, $n \times AEI = 5C + A$. Therefore the Cords of $n \times AE$, $n \times AEG$, $n \times AEI$, are negative, whence by Prop. XXX.

- $x^n - nx^{n-2} \mathcal{E}c. - 2 = -a$, when $\frac{1}{2}n$ is odd.
 and $x^n - nx^{n-2} \mathcal{E}c. + 2 = -a$, when $\frac{1}{2}n$ is even.

that is $x^n - nx^{n-2} \mathcal{E}c. + a \mp 2 = 0$, as $\frac{1}{2}n$ is odd or even. Wherein the Squares of BE, BG, BI, the even Cords are the Roots, and therefore their Product is $a \mp 2$ or rather $2r \mp a \times r^{n-1}$, according as $\frac{1}{2}n$ is odd or even.

14. *Cor. 1.* The same Things supposed; the Sum of the Squares of the odd Cords, as also the Sum of the Squares of the even Cords, $BD^2 + BF^2 + BH^2$ or $BE^2 + BG^2 + BI^2 = n \times CB^2$.

For the Coefficient of the second Term in both Equations is n or nr^2 , the Sum of the Roots.

Cor. 2. The Product of the Squares of the odd Cords + the Product of the Squares of the even Cords, that is, $BD^2 \times BF^2 \times BH^2 + BE^2 \times BG^2 \times BI^2 = 4CB^n$.

16. *Cor. 3.* Hence also if the Circumference be divided into an even Number of equal Parts as n ; and Cords be drawn from any Point B to all the rest, then the Sum of the Squares of the alternate Cords will be equal

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qual to each other and to $n \times CB^2$; $BA^2 + BF^2 + BH^2 + BL^2 + BN^2 = BE^2 + BG^2 + BM^2 + BO^2 = n \times CB^2$. FIG. 16.

This follows from Cor. 1. supposing the Point G to coincide with B, and D with A in Fig. 14, 15.

Cor. 4. If the Arch AR the multiple of AD, be greater than half the Circumference; then the Cord of the Supplement will be $-a$. In which Case you must write $-BR$ for $+BR$, and $+BR$ for $-BR$, in the present Proposition.

SCHOLIUM.

Since the Square of the Cord = versed Sine $\times$ Diameter, therefore, in the foregoing Propositions, and their Corollaries, you may, by Substitution, find the Sums or Products of the versed Sines of Arches, instead of the Sums or Products of the Squares of the Cords corresponding to them.

PROP. XLIV.

If the Circumference of a Circle, whose Diameter is AR, and Center O, be divided into any Number of equal Parts as n , at the Points A, C, E, G &c. beginning at A; and from any Point P in the Diameter AR, be drawn the Lines PC, PE, PG &c. I say the Product of all the Lines $AP \times PC \times PE \times PG$ &c. thro' the whole Circle is $= AO^n - OP^n$. 17 18

For draw the Cords AC, AE, AG; and from C, E &c. let fall Perpendiculars upon the Diameter AR, as CI &c. and let AO or $OR = r$, $PO = x$, $AP = y = r - x$. Then $PC^2 = AP^2 + AC^2 - 2PA \times AI = yy + AC^2 - \frac{2}{r} AC^2$,

that

FIG.

17.

18.

that is $PC^2 = y^2 + \frac{x}{r} AC^2$

likewise $PE^2 = y^2 + \frac{x}{r} AE^2$

also $PG^2 = y^2 + \frac{x}{r} AG^2 \text{ \&c.}$

} these multiply'd

together produce $PC^2 \times PE^2 \times PG^2 = y^6 + \frac{x}{r} y^4 \times$

$\overline{AC^2 + AE^2 + AG^2} + \frac{xx}{rr} y^2 \times \overline{AC^2 \times AE^2 + AC^2 \times AG^2}$

$\overline{+ AE^2 \times AG^2} + \frac{x^3}{r^3} \times \overline{AC^2 \times AE^2 \times AG^2} = PC \times PC \times$

$PE \times PE \times PG \times PG$. But in the second Term of this Equation $AC^2 + AE^2 + AG^2$ is the Sum of the Squares of all the Cords (AC , AE , AG) which call A . And in the third Term is the Sum of all their Rectangles $= B$. In the fourth Term the Sum of all the Solids $= C$, and so on. Likewise the Index of y in the first Term is $n-1$, when n is odd, or $n-2$ when even. This being premised; the Equation will be express'd in general thus, $PC \times PC \times PE \times PE \text{ \&c.} = y^{n-1} + \frac{x}{r} y^{n-3} A + \frac{xx}{rr} y^{n-5} B + \frac{x^3}{r^3} y^{n-7} C + \frac{x^4}{r^4} y^{n-9} D \text{ \&c.}$ when n is odd. And $PC \times PC \times PE \times$

$PE \text{ \&c.} = y^{n-2} + \frac{x}{r} y^{n-4} A + \frac{xx}{rr} y^{n-6} B + \frac{x^3}{r^3} y^{n-8} C \text{ \&c.}$

when n is even. Let $x =$ Cord of AC , $AE \text{ \&c.}$ then,

C A S E I.

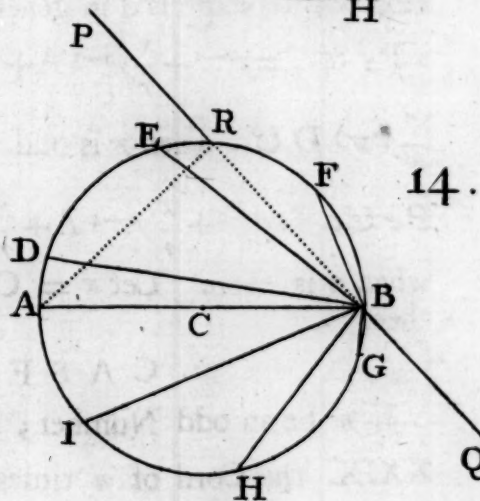
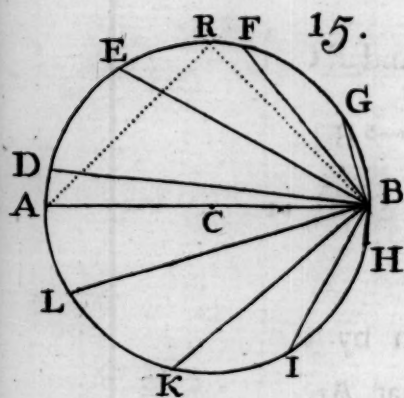
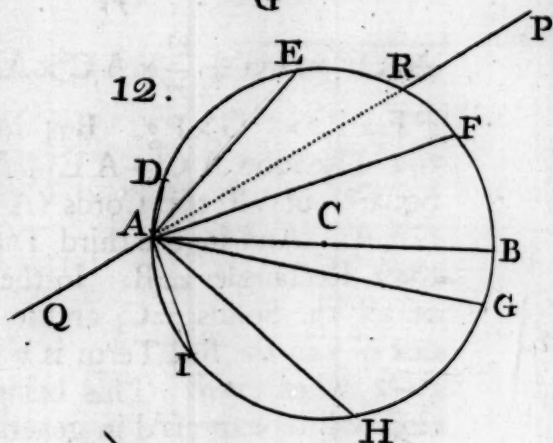
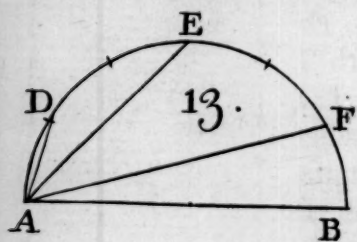
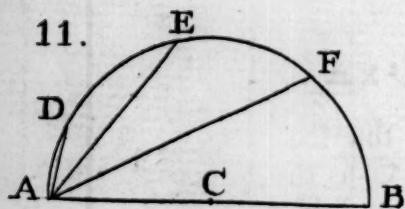
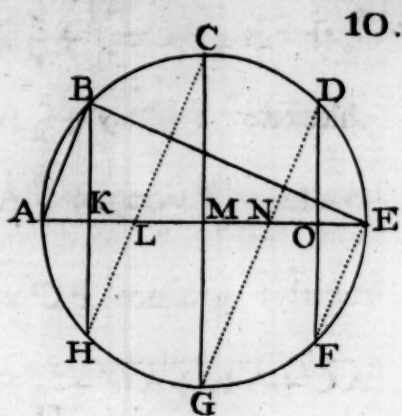
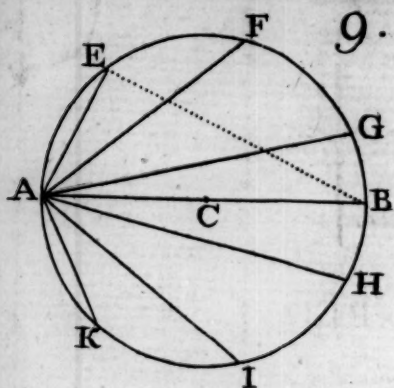
17. If n be an odd Number; then by Cor. 1. Prop.

XXIX. the Cord of n times that Arch $= \frac{x^n}{r^{n-1}} -$

$\frac{nx^{n-2}}{r^{n-3}} \text{ \&c.} = 0$ in this Case. But here one Root is $x=0$,

by which dividing the Equation, and multiplying by r^{n-1} ,





Sect.

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r^{n-1} , you will have $x^{n-1} - nr^2 x^{n-3} + \frac{n \cdot n-3}{2} r^4 x^{n-5} -$

$\frac{n \cdot n-4 \cdot n-5}{6} r^6 x^{n-7} \&c. = 0$, wherein the Roots are

AC, AE, AG, Ag, Ae, Ac. But since AC = Ac, AE = Ae, AG = Ag, and since in this Equation there are only the even Powers of x ; therefore xx is the Algebraic Root of it, and all the particular Roots are AC², AE², AG².

But in every Equation the Coefficient of the second Term is = the Sum of all the Roots, therefore AC² + AE² + AG² = $nr^2 = A$, the Coefficient of the third Term = Sum of all the Rectangles = $\frac{n \cdot n-3}{2} r^4 = B$, the Coefficient of the fourth Term = Sum of all the Solids = $C = \frac{n \cdot n-4 \cdot n-5}{6} r^6$, and so on.

Therefore putting for A, B, C &c. these their Values, and we shall have PC × PE × PG × Pg &c. =

$$y^{n-1} + \frac{x}{r} y^{n-3} \times nr^2 + \frac{xx}{rr} y^{n-5} \times \frac{n \cdot n-3}{2} r^4 \&c. = y^{n-1} + nrxy^{n-3} + \frac{n \cdot n-3}{2} r^2 x^2 y^{n-5} + \frac{n \cdot n-4 \cdot n-5}{6} r^3 x^3 y^{n-7} \&c. \text{ that is, (putting } r=x \text{ for } y \text{ and involving)}$$

$$\begin{aligned} &= r^{n-1} - \frac{n-1}{1} r^{n-2} x + \frac{n-1 \cdot n-2}{2} r^{n-3} x^2 - \frac{n-1 \cdot n-2 \cdot n-3}{6} r^{n-4} x^3 \&c. \\ &+ nr^{n-2} x - \frac{n \cdot n-3}{1} r^{n-3} x^2 + \frac{n \cdot n-3 \cdot n-4}{2} r^{n-4} x^3 \&c. \\ &+ \frac{n \cdot n-3}{2} r^{n-3} x^2 - \frac{n \cdot n-3 \cdot n-5}{2} r^{n-4} x^3 \&c. \\ &+ \frac{n \cdot n-4 \cdot n-5}{6} r^{n-4} x^3 \&c. \\ &\&c. \dots + x^{n-1} \\ &= r^{n-1} + r^{n-2} x + r^{n-3} x^2 + r^{n-4} x^3 \&c. \\ &\&c. \dots + x^{n-1} \end{aligned}$$

then

FIG. then multiply by PA or $r-x$, and you will have
 $PA \times PC \times PE \times PG \times Pg \times Pe \times Pc = r^n - x^n$.

C A S E II.

18. If n be an even Number; then by Cor. 1. Prop.

XXIX. the Cord of n times the Arch $= \sqrt{4rr-xx} \times$

$\frac{x^{n-1}}{r^{n-1}} - \frac{n-2 \cdot x^{n-3}}{r^{n-3}} \mathcal{E}c. = 0$, in which is one Root

$x=0$, and dividing the Equation by x and the correspondent Cord of its Supplement $\sqrt{4rr-xx}$ (which here is AR), and multiplying by r^{n-1} , you will have

$$x^{n-2} - \frac{n-2}{1} r^2 x^{n-4} + \frac{n-3 \cdot n-4}{2} r^2 x^{n-6} -$$

$$\frac{n-4 \cdot n-5 \cdot n-6}{6} r^6 x^{n-8} \mathcal{E}c. = 0$$
, in which there be-

ing only the even Powers of x , the Roots are AC², AE², AG².

But in this Equation the Coefficient of the second Term is = Sum of all the Roots AC² + AE² + AG² = $n-2 \times r^2 = A$, the Coefficient of the 3^d Term is the Sum

of all their Rectangles = $\frac{n-3 \cdot n-4}{2} r^4 = B$. The Co-

efficient of the fourth Term is the Sum of all the Solids = $\frac{n-4 \cdot n-5 \cdot n-6}{6} r^6 = C$, and so on. There-

fore putting for A, B, C $\mathcal{E}c$. their Values in the general Equation, we have PC $\times$ PE $\times$ PG $\times$ Pg $\mathcal{E}c$.

$$= y^{n-2} + \frac{x}{r} y^{n-4} \times \frac{n-2}{r^2} + \frac{x^2}{r^2} y^{n-6} \times \frac{n-3 \cdot n-4}{2} r^4 \mathcal{E}c.$$

$$= y^{n-2} + \frac{n-2}{1} r x y^{n-4} + \frac{n-3 \cdot n-4}{2} r^2 x^2 y^{n-6} +$$

$$\frac{n-4 \cdot n-5 \cdot n-6}{6} r^3 x^3 y^{n-8} \mathcal{E}c. \text{ that is (putting } r-x$$

for y , and involving)

$$= r^n - x^n$$

$$\begin{aligned}
&= r^{n-2} - \frac{n-2}{2} r^{n-3} x + \frac{n-2 \cdot n-3}{2} r^{n-4} x^2 - \frac{n-2 \cdot n-3 \cdot n-4}{6} r^{n-5} x^3 \\
&\quad + \frac{n-2 \cdot n-3 \cdot n-4}{24} r^{n-6} x^4 - \frac{n-2 \cdot n-3 \cdot n-4 \cdot n-5}{120} r^{n-7} x^5 \mathcal{E}c. \\
&\quad + \frac{n-3 \cdot n-4}{2} r^{n-4} x^2 - \frac{n-3 \cdot n-4 \cdot n-5}{6} r^{n-5} x^3 \mathcal{E}c. \\
&\quad + \frac{n-4 \cdot n-5 \cdot n-6}{6} r^{n-6} x^4 \mathcal{E}c. \\
&\quad \mathcal{E}c. \dots + x^{n-2} \\
&= r^{n-2} * \frac{r^{n-4} x^2}{\mathcal{E}c. \dots + x^{n-2}} + r^{n-6} x^4 \mathcal{E}c.
\end{aligned}$$

multiply by $PA \times PR$ or $rr - xx$, and you will have,
 $PA \times PC \times PE \times PG \times PR \times Pg \times Pe \times Pc = r^n - x^n$.
 Q. E. D.

Cor. 1. If the Semi-circumference AER be divided into any Number of equal Parts as n ; at the Points A, B, C, D &c. and from any Point P in the Diameter AR be drawn Lines to all the even Points of Division, as PC, PE, PG. I say, the Product of PA into that of all the Squares of these Lines, $PA \times PC^2 \times PE^2 \times PG^2$, if n is an odd Number, or of PA and PR into these Squares, $PA \times PR \times PC^2 \times PE^2 \times PG^2$ if n is even, is $= AO^n - OP^n$.

For if $n \times AC =$ whole Circumference, then $n \times AB =$ half the Circumference, and $Pc = PC$, $Pe = PE$ &c.

Cor. 2. If a Semi-circle be divided into n equal Parts, and one of them be K, and s, t, u &c. be put for the Cofines of $2K, 4K, 6K$ &c. to $\frac{n-1}{2} K$, if n is odd, or to $\frac{n-2}{2} K$ if even; then I say $r - x \times rr - 2sx + xx \times rr - 2tx + xx \times rr - 2ux + xx$, &c. if n be an odd Number, or $r - x \times r + x \times rr - 2sx + xx \times rr - 2tx + xx \times rr - 2ux + xx$ &c. (if n be even) $= r^n - x^n$.

G

For

FIG.
18.

FIG. For let a, b, c &c. be the Sines of $2K, 4K, 6K$ &c. or of AC, AE, AG &c. then $CI=a, OI=s, PI=s-x, AP=r-x, PR=r+x$, and $PC^2=aa+s-s-x=aa+ss-2sx+xx=rr-2sx+xx$, after the same manner $PE^2=rr-2tx+xx$, and $PG^2=rr-2ux+xx$, &c. and by Cor. 1. the Product of these into $r-x$, or else into $r-x \times r+x$, is $=r^n-x^n$. But observe that the Cosines of Arches greater than 90° , will be negative.

Cor. 3. Hence also if d, e, f &c. be the Sines of $1K, 2K, 3K$, &c. then $xy^2 + \frac{4ddx}{r}xy^2 + \frac{4eex}{r}xy^2 + \frac{4ffx}{r}$ &c. if n be odd,

Or $y \times r+x \times y^2 + \frac{4ddx}{r}xy^2 + \frac{4eex}{r}xy^2 + \frac{4ffx}{r}$ &c. if n be even, $=r^n-x^n$.

This follows from the Demonstration of this Prop. for $2d=AC, 2e=AE, 2f=AG$, &c.

Cor. 4. Also if a, b, c be the versed Sines of $2K, 4K, 6K$ &c.

then if n be odd, y if n be even $y \times r+x$ } $\times y^2 + 2axxy^2 + 2bxxy^2 + 2cx$, &c. $=r^n-x^n$. For $a=\frac{2dd}{r}, b=\frac{2ee}{r}, c=\frac{2ff}{r}$.

17. Cor. 5. And if P be taken without the Circle at π ,
18. the Product of all the Lines $\pi A \times \pi C \times \pi E \times \pi G$ &c. $=\pi O^n - AO^n$.

For let $O\pi : OA :: OA : OP$, and let the Lines PC, PE &c. be drawn; then since $\pi O : CO :: CO : OP$, and Angle O common, therefore the Triangles πOC and COP are similar, and for the same Reason, πEO , and EOP , &c. Whence $\pi O : CO :: \pi C : PC :: \pi E : PE$ &c. $:: \pi R : PR :: \pi A : PA$. Therefore $PA = \frac{r \times \pi A}{\pi O}, PR = \frac{r \times \pi R}{\pi O}, PC = \frac{r \times \pi C}{\pi O}, PE = \frac{r \times \pi E}{\pi O}$ &c. $PO = \frac{rr}{\pi O}$. Whence $AP \times PC$

$\times PE$

$$\times PE \times PG \text{ \š } = \frac{\pi A \times \pi C \times \pi E \times \pi G \text{ \š }}{\pi O^n} r^n = r^n - \text{FIG.}$$

$$\frac{r^{2n}}{\pi O^n}, \text{ or } \pi A \times \pi C \times \pi E \text{ \š } = \pi O^n - r^n.$$

P R O P. XLV.

If the Circumference of a Circle A E G, whose Diameter is A R, and Center O, be divided into any Number of equal Parts as n , at the Points B, D, F &c. and the Part B b be bisected by the Diameter A R; and from any Point P therein there be drawn the Lines P B, P D, P F &c. I say the Product of all these Lines P B $\times$ P D $\times$ P F &c. thro' the whole Circle is $= A O^n + O P^n$. 17.
18.

For let all the Parts B D, D F &c. be bisected in C, E, &c. and P C, P E &c. drawn, and then the Circle will be divided into $2n$ Parts, therefore

$$\text{by the last Prop. } \left\{ \begin{array}{l} P A \times P B \times P C \times P D \times P E \text{ \š } \\ P A \times P C \times P E \text{ \š } \end{array} \right. = A O^{2n} - O P^{2n}$$

$$\text{and by the same, } \left\{ \begin{array}{l} P A \times P C \times P E \text{ \š } \\ P B \times P D \times P F \times P R \times P f \times P d \times P b \end{array} \right. = A O^n - O P^n$$

$$\text{therefore by Division } \left\{ \begin{array}{l} P B \times P D \times P F \times P R \times P f \times P d \times P b \\ P B \times P D \times P F \times P H \times P h \times P f \times P d \times P b, \end{array} \right. \text{ when } n \text{ is odd,}$$

$$\text{or } \frac{A O^{2n} - O P^{2n}}{A O^n - O P^n} = A O^n + O P^n, \text{ when } n \text{ is even, is}$$

Cor. 1. If the Semi-circumference, A L R be divided into any Number of equal Parts as n , in the Points A, B, C, D &c. and from any Point P in the Diameter A R, the Lines P B, P D, P F, &c. be drawn to all the odd Points of Division: Then, the Product of all their Squares if n is even, $P B^2 \times P D^2 \times P F^2 \times P H^2$; or that Product $\times d$ by P R that is $P B^2 \times P D^2 \times P F^2 \times P R$ if n is odd, is $= A O^n + O P^n$.

For if $n \times B b =$ whole Circumference, then $n \times A B = \frac{1}{2}$ Circumference, and $P b = P B$, $P d = P D$ &c.

FIG. Cor. 2. If a Semi-circle be divided into any Number of equal Parts as n , and one of them be K , and
 17. s, t, u &c. be the Cosines of $1K, 3K, 5K$ &c. to
 18. $n-1.K$ when n is even, or to $n-2.K$ when n is odd. Then I say

$$rr - 2sx + xx \times rr - 2tx + xx \times rr - 2ux + xx \text{ \&c.} = r^n + x^n, \text{ when } n \text{ is even, or}$$

$$r + xx \times rr - 2sx + xx \times rr - 2tx + xx \times rr - 2ux + xx \text{ \&c.} = r^n + x^n, \text{ when } n \text{ is odd.}$$

For if a, b, c &c. be the Sines of $1K, 3K, 5K$ &c. then the same way as in Cor. 2. Pr. XLIV. will be found $PB^2 = rr - 2sx + xx$, $PD^2 = rr - 2tx + xx$, $PF^2 = rr - 2ux + xx$ &c. whose Product alone, or else multiply'd by $r+x$, is, by Cor. 1. $= r^n + x^n$.

If P be without the Circle as at π , it will still be $\pi B \times \pi D \times \pi F$ &c. $= A O^n + O \pi^n$.

For by this Prop. and Cor. 5. of the last; that Product will be $\frac{O \pi^{2n} - A O^{2n}}{O \pi^n - A O^n} = O \pi^n + A O^n$.

P R O P. XLVI.

20. If the Arch Ae of a Circle be divided into any Number of equal Parts as n , of which AB is one; and if the whole Circumference be divided into the same Number of equal Parts, at the Points B, C, D, E &c. beginning at B ; and if from any Point P , taken in the Diameter AR , the Lines PB, PC, PD &c. be drawn, and eF be let fall perpendicular to AR to intersect it in F , between the Center O and R . And if $AO = r$, $OP = x$, $OF = b$; then I say the Product of all the Squares $PB^2 \times PC^2 \times PD^2 \times PE^2$ &c. $= r^{2n} + 2br^{n-1}x^n + x^{2n}$.

For let $AO = r = 1$. $AP = y = 1 - x$, then you will find as in Prop. XLIV. $PB^2 = y^2 + xx \times AB^2$, $PC^2 = y^2 + xx \times AC^2$, $PD^2 = y^2 + xx \times AD^2$ &c. which multiply'd together, produce $PB^2 \times PC^2 \times PD^2$ &c. $= y^n + Axy^{n-2} + Bx^2y^{n-4} + Cx^3y^{n-6}$ &c. $+ Fx^n$, putting $n = 2n$, $A =$ Sum of the Squares of the Cords $AB^2 + AC^2 + AD^2$ &c.

$\mathcal{E}c$. B = Sum of their Rectangles, C = Sum of their Solids $\mathcal{E}c$. And F = Product of them all. FIG. 20.

Now if z be the Cord of an Arch, c the Cord of n times the Arch, then by Prop. XXIX. $z^n - \nu z^{n-2} + \nu$.

$$\frac{\nu-3}{2} z^{n-4} - \nu \cdot \frac{\nu-4 \cdot \nu-5}{2 \cdot 3} z^{n-6} \mathcal{E}c. = \pm cc.$$

In which Equation, the Root zz represents indifferently the Square of any Cord A B, A C, A D $\mathcal{E}c$. Therefore by common Algebra, the Sum of the Squares of the Cords, $A B^2 + A C^2 + A D^2 \mathcal{E}c. = \nu$, the Sum of their Products $= \nu \cdot \frac{\nu-3}{2}$; Sum of their

Solids $= \nu \times \frac{\nu-4 \cdot \nu-5}{2 \cdot 3} \mathcal{E}c$. and the Product of all

$= cc$. Whence we get $A = \nu$, $B = \frac{\nu \cdot \nu-3}{2}$, $C =$

$\frac{\nu \cdot \nu-4 \cdot \nu-5}{2 \cdot 3} \mathcal{E}c$. and $F = cc$. Consequently $P B^2 \times$

$P C^2 \times P D^2 \mathcal{E}c. = y^n + \nu x y^{n-2} + \nu \cdot \frac{\nu-3}{2} x^2 y^{n-4} + \frac{\nu \cdot \nu-4 \cdot \nu-5}{2 \cdot 3} x^3 y^{n-6} \mathcal{E}c. + cc x^n$, and putting $1-x$ for y , and involving, we have $P B^2 \times P C^2 \times P D^2 \mathcal{E}c. =$

$$\begin{aligned} 1 - \nu x + \nu \cdot \frac{\nu-1}{2} x^2 - \nu \cdot \frac{\nu-1}{2} \cdot \frac{\nu-2}{3} x^3 \mathcal{E}c. \dots \mathcal{E}c. - \nu x^{n-1} + x^n \\ + \nu x - \nu \cdot \frac{\nu-2}{1} x^2 + \nu \cdot \frac{\nu-2}{1} \cdot \frac{\nu-3}{2} x^3 \mathcal{E}c. \quad \mathcal{E}c. + \nu x^{n-1} \\ + \nu \cdot \frac{\nu-3}{2} x^2 - \nu \cdot \frac{\nu-3}{2} \cdot \frac{\nu-4}{1} x^3 \mathcal{E}c. \\ \mathcal{E}c. \\ + cc x^n \end{aligned}$$

Hence it will appear, by summing them up, that the Coefficients of the second, third $\mathcal{E}c$. Terms are 0, till you come to the middle Term, where you have $cc x^n$ (the Index n being $\frac{1}{2} \nu$); likewise beyond the middle Term they are 0, because the Coefficients are

FIG. the same at equal Distances on each Side from the Middle; whence they all vanish till the last x^n .

20.

Now to find the Coefficient of the middle Term, it must be observed that ccx^n supplies the Place of the first Term of the next horizontal Series; which Term, if it were there, then the Sum of the Coefficients (in that perpendicular Column) would there also be $=0$, as before. Hence it follows that if $v=2$,

4, 6 &c. that the Term $v x$, $v \cdot \frac{v-3}{2} x^2$, $v \cdot \frac{v-4}{2} \cdot \frac{v-5}{3} x^3$ &c. will be wanting, respectively, which Term is $2x$, $2xx$, $2x^3$, &c. respectively, and in general the

Term wanting is always $\frac{1 \cdot 2 \cdot 3 \dots n-1 \times v}{1 \cdot 2 \cdot 3 \dots n-1 \cdot n} x^n = \frac{v}{n} x^n$

$=2x^n$. Consequently the middle Term will be found to be $-2x^n + ccx^n$. But $cc = AR \times AF = 2+2b$, therefore $-2x^n + 2+2b \cdot x^n = 2bx^n$ is the middle Term. And therefore $P B^2 \times P C^2 \times P D^2$ &c. thro' the whole Circle, $= 1+2bx^n + x^{2n}$, or rather $r^{2n} + 2r^{n-1}bx^n + x^{2n}$.

21.

Cor. 1. If the Arch Ae be less than a Quadrant, and F lye from O towards A ; then will $P B^2 \times P C^2 \times P D^2 \times P E^2$ &c. $= r^{2n} - 2br^{n-1}x^n + x^{2n}$.

This follows from this Prop. putting $-b$ instead of $+b$, as lying on the contrary Side of O .

20.

Cor. 2. If P be without the Circle as at π , then will $\pi B^2 \times \pi C^2 \times \pi D^2$ &c. $= r^{2n} \pm 2br^{n-1}x^n + x^{2n}$, ac-

21.

cording as F falls beyond, or on this Side of O , in respect to P .

The Demonstration of this Prop. equally proves this, since the even Powers of $x-1$, are the same with those of $1-x$, and v or $2n$ is an even Number.

21.

Cor. 3. Let $b = \text{Cosine of an Arch } A \text{ or } Ae$, to the Radius r ; and $\frac{A}{n} = B$, $\frac{360}{n} = C$, and s, t, u &c. the Cosines of $B, B+C, B+2C, B+3C$, &c. to $B + \frac{n-1}{n}C$,

$n-1$. C, to the Radius r , x = any Quantity : Then I FIG. 21.
 say, $rr-2sx+xx \times rr-2tx+xx \times rr-2ux+xx \text{ \&c.}$
 to n Terms, is $=r^{2n}-2br^{n-1}x^n+x^{2n}$.

For let $a=BI$ the Sine of B , $s=OI$ the Cosine,
 then $PB^2=BI^2+PI^2=aa+s-x^2=aa+ss-2sx+xx$
 $=rr-2sx+xx$, for the same Reason, $PC^2=rr-2tx$
 $+xx$, $PD^2=rr-2ux+xx$, &c. but by Cor. 1. the
 Product of these is $=r^{2n}-2br^{n-1}x^n+x^{2n}$.

If A be greater than a Quadrant, write $+b$ for $-b$.
 Likewise any of the Cosines s , t , u &c. of Arches
 between 90 , and 270 , is negative, and therefore in
 that Case write $+s$, $+t$, $+u$ instead of $-s$, $-t$, $-u$,
 &c.



THE
ELEMENTS
OF
TRIGONOMETRY.

B O O K II.

*Plain Trigonometry, or the Doctrine of plain
Triangles.*

PLAIN *Trigonometry* teaches the Relation of the Sides and Angles of Plain Triangles, and how to calculate their Sides and Angles.

A *Plain Triangle* is made by three right Lines, which are called it's Sides, and if one Side be perpendicular to another, or makes a right Angle with it; then 'tis call'd a *right Angle-Triangle*, otherwise 'tis an oblique Triangle.

A *right Angle* is measured by an Arch of 90° , an *acute Angle* is less, and an *obtuse* one greater than a right Angle.

In a right Angled-triangle, the Side opposite to the right Angle is call'd the *Hypotenuse*; and the other two the *Legs* or *Sides*, or sometimes the *Perpendicular* and *Base*.

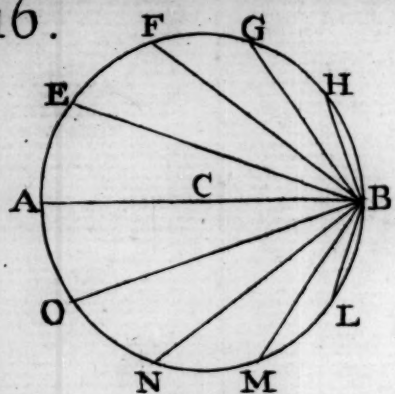
In all plain Triangles the Sum of the three Angles together is 180 Degrees: And in a right Angle-triangle, the Sum of the two oblique Angles is 90° .

When three Letters denote an Angle, the middle Letter is always at the Angle.

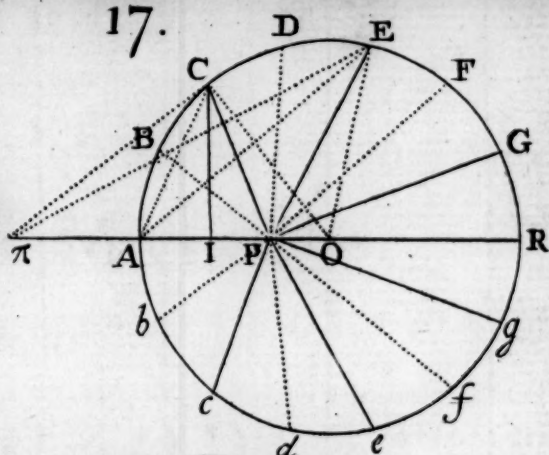
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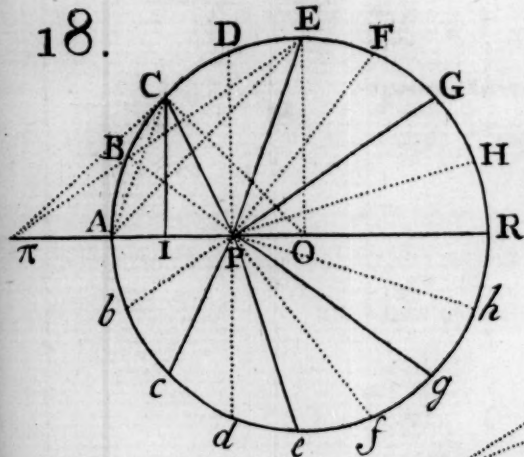
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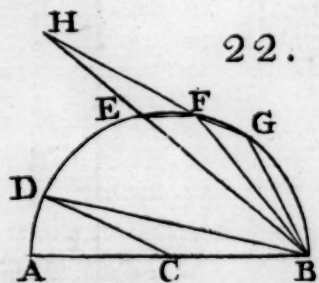
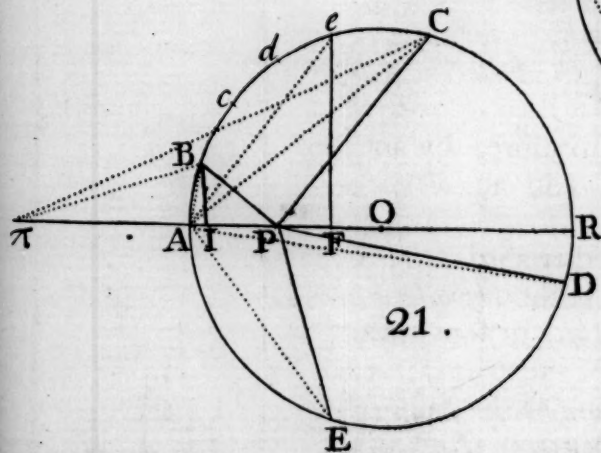
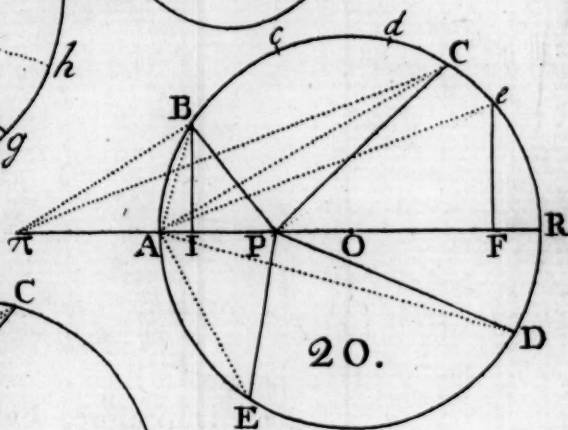
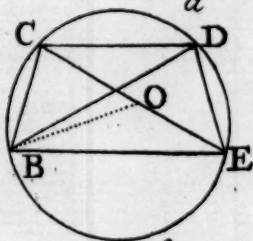
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18.



19.



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S E C T. I.

The Relations, and several Proportions among the Sides, and Angles of plain Triangles.

P R O P. I.

In any right angled Triangle CAB, let the Hypotenuse $CA = b$, and $CA - CB$ or $BD = v$; then, FIG. 1.
 $57.29578 \sqrt{\frac{2v}{b}} + \frac{v}{3.4b} A + \frac{3^2v}{5.8b} B + \frac{5^2v}{7.12b} C, \&c. =$
 Angle ACB in Degrees; where A, B, C are the preceding Terms.

For with the Center C describe the Arch AD, then BD or v is the versed Sine of the Arch AD, and Radius b , therefore (by Prop. XIV. B. I) Arch AD = $\sqrt{2bv} + \frac{v}{3.4b} A + \frac{3^2v}{5.8b} B + \frac{5^2v}{7.12b} C \&c.$ but as half the Circumference $3.14159b : 180 \text{ Degrees} :: AD : \frac{180 AD}{3.14159} = \text{Degrees in the Arch AD} = \frac{57.295779}{b} \times AD$, that is, $\angle ACD = \frac{57.29}{b} \times \sqrt{2bv} + \frac{v}{3.4b} A : \&c.$
 $= 57.2957 \sqrt{\frac{2v}{b}} + \frac{v}{3.4b} A + \frac{3^2v}{5.8b} B \&c.$

Cor. 1. As the Sum of the Hyp. and $\frac{1}{2}$ longer Leg, $AC + \frac{1}{2}CB$:

To 86 ::

So the shorter Leg, AB :

To its opposite Angle C, nearly.

Let $CB = b$, and $AB = c$, $a = \angle C$ in Degrees, then by the Nature of the Circle $c = \sqrt{2bv - vv} = \sqrt{2bv - \frac{v^2}{4b}}$ by Evolution. Then by this Prop. $a =$

FIG.

$\frac{57.3}{b} \times : \sqrt{2bv} + \frac{v}{12b} \sqrt{2bv} : \text{nearly, multiply this E-}$
 quation by the former, and you have $a \times : \sqrt{2bv} -$
 $\frac{v}{4b} \sqrt{2bv} = \frac{57.3^c}{b} \times : \sqrt{2bv} + \frac{v}{12b} \sqrt{2bv}$, and dividing
 by $\sqrt{2bv}$, $a \times 1 - \frac{v}{4b} = \frac{57.3^c}{b} \times 1 + \frac{v}{12b}$. And $\frac{57.3^c}{b}$

$$= a \times \frac{1 - \frac{v}{4b}}{1 + \frac{v}{12b}} = (\text{by Division}) a \times 1 - \frac{v}{3b} \quad \&c.$$

$$= a \times \frac{3b - v}{3b} = a \times \frac{3b - b + b}{3b} = a \times \frac{2b + b}{3b}, \text{whence}$$

$$a \times 2b + b = 171.9^c = 172^c \text{ nearly, or } b + \frac{1}{2}b \times a = 86^c.$$

S C H O L I U M.

If $c = 1$. $n = 57.3$, then $\frac{1}{2}n = 86$, and by Cor. 1.
 $b + \frac{1}{2}b : 1 :: \frac{1}{2}n : a$, and $2b + b = \frac{3n}{a}$, and $b = \frac{3n}{2a} - \frac{b}{2}$
 $= \frac{3n}{2a} - \frac{\sqrt{bb} - 1}{2}$, whence $\frac{3n}{a} - 2b = \sqrt{bb} - 1$. this E-
 quation reduced gives, $b = \frac{2n}{a} - \sqrt{\frac{nn}{aa} - \frac{1}{3}}$. Whence
 $b = \frac{3n}{a} - 2b$; which is *Forster's Rule*.

P R O P. II.

1. If any right angled Triangle CAB, let $a = \text{Degrees}$
 in the lesser $\angle C$, $m = \frac{a}{57.29578}$. Then

As 1 :

$$\text{To } 1 - \frac{m^2}{1.2} - \frac{m^2}{3.4} A - \frac{m^2}{5.6} C - \frac{m^2}{7.8} D \quad \&c. ::$$

As

SECT. I. of TRIGONOMETRY.

91
FIG.
1.

As the Hypotenuse, CA :

To the adjoining Side CB.

where A, B, C &c. are the foregoing Terms, with their Signs.

For let Radius CA or CD = b . Arch AD = z , then

$$z = \frac{ab}{57.295 \text{ } \mathcal{E}c.} = mb, \text{ and by Cor. 1. Prop. XII. Co-}$$

$$\text{fine CB} = b - \frac{z^2}{1.2b} + \frac{z^4}{1.2.3.4b^3} \mathcal{E}c. = b - \frac{m^2b^2}{1.2b} + \frac{m^4b^4}{1.2.3.4b^3} - \mathcal{E}c. = CA \times 1 - \frac{m^2}{1.2} + \frac{m^4}{1.2.3.4} - \frac{m^6}{1...6} + \frac{m^8}{1...8} \mathcal{E}c.$$

Cor. 1. If a be the lesser Angle, in Degrees, then it will be,

$$\text{As } \frac{57.3}{a} + \frac{3a}{1000} : 1 :: \text{Hyp. CA} : \text{op. side AB, nearly.}$$

$$\text{For let } n = 57.3, \text{ then } CB^2 = b^2 - z^2 + \frac{z^4}{3b^2} \mathcal{E}c.$$

$$\text{and } AB^2 = CA^2 - CB^2 = z^2 - \frac{z^4}{3b^2}, \text{ and extracting the}$$

$$\text{Root, } AB = z - \frac{z^3}{6bb} = mb - \frac{m^3b}{6} = b \times \frac{a}{n} - \frac{a^3}{6n^3}; \text{ and}$$

$$\text{dividing both Sides by } \frac{a}{n} - \frac{a^3}{6n^3}, b = AB \times \frac{n}{a} + \frac{a}{6n} =$$

$$: \frac{57.3}{a} + \frac{3a}{1031} \times : AB.$$

Cor. 2. If a be the lesser Angle in Degrees, then

$$\text{As } 1 : 1 - \frac{1\frac{1}{2}aa}{10000} :: \text{Hyp. CA} : \text{adjoining Side BC, nearly.}$$

$$\text{For } CB = CA \times 1 - \frac{m^2}{2} = CA \times 1 - \frac{aa}{2nn} = CA \times$$

$$1 - \frac{1.52aa}{10000}.$$

SCHO-

FIG.

S C H O L I U M.

- I. Let $q = 90^\circ$, $y = q - a$, then since $CB = CA \times 1 - \frac{aa}{2nn}$, by Division $CA = CB \times 1 + \frac{aa}{2nn}$, and $CA \times y = CB \times y + \frac{4aa}{8nn}$. But $\frac{8nn}{y} = \frac{8nn}{q-a} = \frac{8nn}{q} + \frac{8na}{qq}$

$$\text{Ec.} = 2q2 + 3a. \text{ Therefore } CA \times y = CB \times y + \frac{4aa}{300 + 3a},$$

nearly, which is *Wilson's Rule*.

Since in a right Angled-triangle, the Square of the Hypothenufe is equal to the Sum of the Squares of the Sides. And in any Triangle, the Base is to the Sum of the Sides, as their Difference to the Difference of the Segments made by a Perpendicular. Therefore by these and the two foregoing Propositions and their Corollaries, all the Cases of plain Triangles may be resolved, without any Tables whatever: Dividing oblique Triangles into two right angled ones, by a Perpendicular let fall from the End of a given Side and opposite to a given Angle.

P R O P. III.

- I. In any right angled Triangle CAB , any one of the three Sides being made a Radius; and a Circle described from the End of it as a Center; then the other Sides will represent Sines, Tangents, Secants, &c. of the respective Angles: Then it will be as any Side; to what it represents in the Table :: so is any other Side, to what it represents in the Table.

Describe the Arches AD and BE , from the Center C , then if CA be made Radius; AB will be the Sine, and CB the Cosine, of the Angle C . And if CB be made Radius, AB will be the Tangent, and CA the Secant of the Angle C . And because the

Sines,

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Sines, Tangents &c. of any Angles may be found in FIG. the Tables, therefore when any Triangle is proposed, there may always be found a similar one in the Tables; and consequently the Sides about the equal Angles are proportional. Therefore it will be,
 As Radius : AC :: S. C : AB :: Cof. C : CB,
 and Rad. : CB :: Tan. C : AB :: Sec. C : CA.

PROP. IV.

In any right angled Triangle CDE,
As the Sum of the Hyp. and one Side, CE+ED :
Radius ::
So the other Side, CD :
To Tan. half the op. Angle CED.

Produce DE to B, so that EB=EC, then $\angle B = \angle BCE$, and Angle DEC = $\angle B + \angle BEC = 2\angle B$. But in the Triangle CDB, by Prop. III. it is as DB : Radius :: CD : Tan. B, that is DE+EC : Rad. :: CD : Tan. $\frac{1}{2}E$.

PROP. V.

The Sides of any Triangle, are proportional to the Sines of their opposite Angles, AB : S.F :: AF : S.B :: FB : S.A.

On AB, AF let fall the Perpendiculars FH, Bb;
 then by Prop. III. $AF : FH :: R : S.A$
 $FH : FB :: SB : R$
 therefore ex equo, $AF : FB :: S.B : S.A$.

Again,

As $AB : Bb :: R : S.A$
 and $Bb : BF :: S.F : R$
 and ex equo $AB : BF :: S.F : S.A$.

Cor. 1. If a Line AD be drawn from the Vertex of any Triangle cutting the Base; the Rectangle of the

FIG. the Sides and Sines of the vertical Angles, are directly as the Segments of the Base.

$$4. \quad \text{For } S.CDA : CA :: S.CAD : CD = \frac{CA \times S.CAD}{S.CDA}.$$

$$\text{And } S.CDA \text{ or } BDA : AB :: S.DAB : DB = \frac{AB \times S.DAB}{S.CDA} \text{ therefore } CD : DB :: CA \times S.CAD : AB \times S.DAB.$$

5. *Cor. 2.* If in a Triangle two Sides AC, CB, and the Angle A opposite to the lesser Side CB, be given; the Angle opposite to the greater may be either less or greater than a right Angle; also the Base and it's opposite Angle, will have each of them, two separate Values. Therefore the three Things required are ambiguous.

For make $CD=CB$, then $\angle CDB = \angle B$, and there are the two Triangles ACB, ACD, wherein two Sides and an opposite Angle are given the same in both; therefore the other three things may be found in either Triangle; for the Angle opposite to AC may be either B or it's Sup. ADC, and the Base may be either AB or AD, and it's op. Angle may be either ACB or ACD.

P R O P. VI.

6.

In any Triangle ABC,

As the Sum of any two Sides $BC+AB$:

To their Difference $BC-AB$::

Tan. half the Sum of their op. Angles, Tan. $\frac{A+C}{2}$:

Tan. half their Difference, Tan. $\frac{A-C}{2}$.

Make $BD=BA$, and draw AD, and $BFG \perp$ to it; likewise $FE \parallel AC$, then will $AF=FD$, $DE=EC$.

E C. The $\angle B A F = \frac{B A D + B D A}{2} = \frac{B A C + C}{2}$, FIG. 6.

and $\angle G A F = B A C - B A F = \frac{B A C - C}{2}$.

The Triangles B F E, B G C are similar, whence
 $B E : E C :: B F : F G$. But $A F : R :: B F : \text{Tan.}$
 $B A F :: F G : \text{Tan. F A G}$, or $B F : F G :: \text{Tan.}$
 $B A F : \text{Tan. F A G}$. Consequently $B E$ or $\frac{B D + B C}{2}$

: $E C$ or $\frac{B C - B D}{2} :: \text{Tan. B A F} : \text{Tan. F A G}$.

Otherwise.

$B C : B A :: S. A : S. C$. and $B C + B A : B C -$
 $B A :: S. A + S. C : S. A - S. C ::$ (by Cor. 1. Pr. VII.)
 $\text{Tan. } \frac{A + C}{2} : \text{Tan. } \frac{A - C}{2}$.

Note. Instead of the Tan. half the Sum of the op. Angles, you may take the Cotangent of half the included Angle, which is equal to it.

Cor. 1. As the Base C A :

Sum of the Sides, $C B + B A ::$

So diff. Sides C D :

Diff. Segments of the Base made by a Perpendicular ::

So Cof. $\frac{1}{2}$ Sum op. Angles, or S. $\frac{1}{2}$ the vertical Angle :

Cof. $\frac{1}{2}$ diff. op. Angles, or Cof. $\frac{1}{2}$ diff. vertical Angles, made by a Perp.

For since $D F = \frac{1}{2} D A$, $F E = \frac{1}{2} A C$; and in the Triangle F B E, $F E : B E :: S. F B D$ or Cof. B D F : S. B F E or G F E or Cof. D F E. But D F E or D A C = B D A - C = B A D - C, and $2 D A C = B A C - C$, also $\frac{1}{2} B = \text{Comp. } \frac{A + C}{2}$, and $A - C =$ diff. vert. $\angle S$ made by a Perpendicular.

Likewise

FIG.

6.

Likewise it is evident from Geometry, that $CA : CB + BA :: CD : \text{diff. Segments of the Base.}$ Supposing a Circle described with Center B and Radius B A. See *Eucl.* III. Prop. XXXVI. and Cor.

Cor. 2. As the Base CA :

Difference of the Sides CD ::

So Sum of the Sides CB + BA :

Diff. Segments by a Perpendicular :

So Sine half Sum op. Angles, $\frac{A+C}{2}$:

Sine of half their Difference, $\frac{A-C}{2}$.

For in the Triangle ADC, $AC : DC :: S.ADB : S.CAD$.

Cor. 3. As Square of the Base AC^2 :

Difference of the Squares of the Sides, $BC^2 - BA^2 ::$

S. Sum op. Angles, $A+C$:

S. their difference, $A-C$.

For by multiplying the two Proportions in Cor. 2

and §. $AC^2 : CB + BA \times CB - BA :: \text{Cof. } \frac{A+C}{2}$

$\times S. \frac{A+C}{2} : \text{Cof. } \frac{A-C}{2} \times S. \frac{A-C}{2} ::$ (by Cor. 3. Pr.

II.) $S.A+C : S.A-C$.

P R O P. VII.

7.

In any Triangle AFB,

As twice the Rectangle of the Legs, $2AF \times FB$:

Sum of Squares of the Legs — Square Base ::

Radius :

Cof. vertical Angle, F.

On FB let fall the Perpendicular AC, then by Prop. XIII. *Eucl.* II. $AB^2 + 2BF \times CF = AF^2 + FB^2$, and $2BF \times FC = AF^2 + FB^2 - AB^2$. But by Prop. III. $AF : FC :: \text{Rad} : \text{Cof. F}$. And multiplying the

the two first Terms by $2FB$, then will $2AF \times FB : FIG.$
 $2BF \times FC$ or $AF^2 + FB^2 - AB^2 :: Rad : Cof. F.$ 7.

Cor. 1. Hence these are respectively proportional,
 As twice the Rectangle of the Legs :
 Rectangle of the Base + diff. Legs, and Base
 — diff. Legs :
 Rectangle of the Sum of the three Sides, and
 Sum Legs — Base ::

So Radius :

Verfed Sine of the included Angle :

Verfed Sine of the Sum of the opposite Angles.

For let $m=AF$, $n=BF$, $b=AB$, then by this
 Prop.

$r : Cof. F :: 2mn : mm + nn - bb$; and by Com-
 position,

$r : r \pm Cof. F :: 2mn : 2mn \pm mm \pm nn \mp bb$,
 that is by Schol. Prop. I.

$r : \left\{ \begin{array}{l} \text{verf. Sup. F} \\ \text{verf. F} \end{array} \right\} :: 2mn : \left\{ \begin{array}{l} m+n^2-bb = m+n+b \times m+n-b \\ bb-m-n^2 = b+m-n \times b-m-n. \end{array} \right.$

Cor. 2. From half the Sum of the three Sides,
 subtract each Side severally, and note the three Re-
 mainders. Then

As half the Rectangle of the Legs :

To Square Root of the Product of this half Sum
 and the three Remainders ::

So Radius :

Sine of the vertical Angle.

For let $s=b+m+n$, $v=\text{verf. F}$. $V=\text{verf. Sup. F}$.
 Then by *Cor. 1.*

$v = \frac{b+m-n \times b+n-m}{2mn} r = \frac{s-2n \times s-2m}{2mn} r$, and $V =$

$\frac{b+m+n \times m+n-b}{2mn} r = \frac{s \times s-2b}{2mn} r$. And by Schol.

Prop. I.

H

$\sqrt{V_0}$

FIG.

7.

$$\sqrt{V_u} = S.F = \frac{r}{2mn} \sqrt{s \times s - 2b \times s - 2m \times s - 2n} = \frac{2r}{mn} \times \sqrt{\frac{s}{2} \times \frac{s-2b}{2} \times \frac{s-2m}{2} \times \frac{s-2n}{2}}.$$

Cor. 3: As the Rectangle of the Legs :

Radius Square ::

So the Rectangle $\frac{1}{2}$ Sum 3 Sides — one Leg, and
that $\frac{1}{2}$ Sum — other Leg :

Sine Square of half the vertical Angle ::

And so the Rectangle $\frac{1}{2}$ Sum 3 Sides, and that
 $\frac{1}{2}$ Sum — Base :

Cofine Square of half the vertical Angle.

For multiplying the three last Terms in Cor. 1. by $\frac{1}{2}r$; then $2mn : b+m-n \times b+n-m : s \times s - b :: \frac{1}{2}rr : \frac{1}{2}rv$. But by Prop. II. $\frac{1}{2}rv =$ Sine Square of half F, and $\frac{1}{2}rV =$ Cof. Square of $\frac{1}{2}F$. Therefore $2mn : s-2m \times s-2n : s \times s - b :: \frac{1}{2}rr : \text{Sine}^2 : \text{Cofine}^2$, or $mn : \frac{s-2m}{2} \times \frac{s-2n}{2} : \frac{s}{2} \times \frac{s-b}{2} :: rr : \text{Sine}^2 : \text{Cofine}^2$, of $\frac{1}{2}F$. And hence also follows.

Cor. 4. As four Times the Rectangle of the Legs :

To Radius Square ::

So Rectangle of the Base + diff. Legs, and Base
— diff. Legs :

Sine Square of half the vertical Angle ::

And so Rectangle of Sum Legs + Base, and Sum
Legs — Base :

Cofine Square of half the vertical Angle.

Cor. 5. Rectangle of Sum of the three Sides, and
Sum Legs — Base :

Rectangle Base + diff. Legs, and Base — diff.
Legs ::

Radius Square :

Tangent Square of half the Angle at the Vertex.

This follows from Cor. 4. because Cofine : Sine ::

Rad : Tangent, by Prop. I. B. I.

SCHOL.

S C H O L I U M.

Instead of the Rectangle of the Legs, you may take a Quarter of the Rectangle, of the Sum + the Difference of the Legs, into the Sum — diff. Legs; which is equal thereto.

Likewise instead of half the vertical Angle, you may take the Complement of half the Sum of the opposite Angles, which is also equal to it.

P R O P. VIII.

In any Triangle AFB, I say

$$AB - \frac{AF \times \text{Cof. } A}{\text{Rad.}} : S.A :: AF : \text{Tan. opposite } \angle B.$$

For let fall the Perpendicular FR on the Base; then in the Triangle AFR, Rad. : AF :: S.A : FR
 $= \frac{AF \times S.A}{r} :: \text{Cof. } A : AR = \frac{AF \times \text{Cof. } A}{r}$. And

$$BR = AB - \frac{AF \times \text{Cof. } A}{r}, \text{ then } BR : \text{Rad.} : FR \text{ or } \frac{AF \times S.A}{r} : \text{Tan. } B. \text{ Or } BR : S.A :: AF : \text{Tan. } B.$$

$$\text{Cor. } \sqrt{AB^2 + AF^2 - \frac{2AB \times AF}{\text{Rad.}} \times \text{Cof. } A} = FB,$$

$$\text{For } FB^2 = AB^2 + AF^2 - 2AB \times AR.$$

P R O P. IX.

In the Triangle AFB, draw FR perpendicular to the Base, and make BR = RD; then

As the Base, AB :

To the Difference of the Segments AD ::

So Sine of the vertical Angle F :

To Sine of the diff. vert. Angles, or S. diff. Angles at the Base B, A.

$$\text{For } AB : S.AFB :: FB \text{ or } FD : S.A :: AD : S.AFD.$$

$$\text{And } \angle B - \angle A = \angle FDB - \angle A = \angle AFD.$$

H 2

PROP.

P R O P. X.

9.

*As Cos. diff. Angles at the Base — Cos. Sum :
Sine of the vertical Angle ::
So Perpendicular :
Half the Base.*

For let ADE be the Triangle, DP the Perpendicular, C the Center of the circumscribing Circle, draw ECG perpendicular, and DF parallel to AB, &c. the $\angle ECB (= 2 EBA)$ is the Sum of the Angles A, B, whose versed Sine is EG. and ECD or 2 EBD is the Difference of the Angles A, B, whose versed Sine is EF; therefore the Difference of these versed Sines, or the Difference of the Cosines is FG or DP. Also GCB or ADB is the vertical Angle, whose Sine is GB. And therefore $DP : GB :: \text{diff. Cosines} : S. \text{vertical Angle}.$

*Cor. As Sine of the diff. Angles at the Base :
Sine of the vertical Angle ::*

*Distance of the Perp. from the Middle of the
Base :*

Half the Base.

For FD is the Sine of the Difference of the Angles, A, B, and FD or GP : GB :: S. diff. : S. vert. Angle.

P R O P. XI.

8.

As Radius :

To Sum of the Cotan. half the Angles at the Base ::

So the Perpendicular :

To Sum of the three Sides.

Put $FR = p$, Radius $= r$, then by Prop. IV. Tan.

$$\frac{1}{2}A : r :: r : \text{Cot. } \frac{1}{2}A :: p : AF + AR$$

$$\text{and } r : \text{Cot. } \frac{1}{2}B :: p : BF + BR$$

$$\text{whence } AF + BF + AB = \frac{\text{Cot. } \frac{1}{2}A + \text{Cot. } \frac{1}{2}B}{r} p.$$

P R O P.

P R O P. XII.

In a right angled Triangle ACB,

As Radius :

Sine of double one acute Angle, A or B ::

Square of the Hypothenufe :

Four times the Area.

FIG.

10.

Draw $CD \perp AB$, and bissect the Hyp. AB in E, and draw EC, then $AE = EC = EB$, because ACB may be inscrib'd in a Semi-circle; then by similar Triangles, $AB : BC :: AC : CD = \frac{AC + BC}{AB}$. Al-

so $\angle CED = \angle B + BCE = 2B$, and $CEB = 2A$.

Therefore by Prop. III. as Rad : CE or $\frac{1}{2} AB :: S.E$

: CD, or Rad : S.E :: $\frac{1}{2} AB : \frac{AC \times CB}{AB} :: \frac{1}{4} AB^2$

: $\frac{AC \times CB}{2}$ or the Area of the Triangle.

P R O P. XIII.

As Radius :

To Sine of any Angle of a Triangle, A ::

So half the Rect. of the including Sides, $\frac{AF \times AB}{2}$:

3.

To the Area of the Triangle.

For the Area of the Triangle AFB is $\frac{1}{2} AB \times FH$; but Rad : AF :: S.A : FH = $\frac{AF \times S.A}{Rad.}$, therefore

the Area = $\frac{AF \times AB \times S.A}{2 Rad.}$.

P R O P. XIV.

As Cos. diff. — Cos. Sum of the Angles at the Base :

Sine of the vertical Angle ::

So the Square of the Perpendicular :

To the Area of the Triangle ::

9.

H 3

And

FIG. 9. *And so the Area of the Triangle :
To the Square of half the Base.*

For by Prop. X. $\text{Cof. } \overline{B-A} - \text{Cof. } \overline{B+A} : \text{S.D.} ::$
 $\text{DP} : \text{GB} :: \text{DP}^2 : \text{GB} \times \text{DP}$, the Area $:: \text{GB} \times \text{DP}$
 $: \text{GB}^2$.

PROP. XV.

3. *In any Triangle,
As Radius :
Tan. half the vertical Angle ::
Rect. of half the Perimeter, and that half — Base :
To the Area of the Triangle.*

For, let $s = \text{S. AFB}$; c, t, τ the Cosine, Tangent, and Cotangent of half AFB, $m = \text{AF}$, $n = \text{FB}$, $\text{AB} = b$, $z = \frac{m+n+b}{2}$. Then by Prop. XIII. $\frac{mns}{2r} =$ Area, but by Sch. Prop. II. B. I. $s = \frac{2cc}{\tau} = \frac{2cct}{rr}$, therefore $\frac{mncct}{r^3} = \text{Area}$. And by Cor. III. Pr. VII. $mnc = z \times \overline{z-b}$, therefore $\frac{z \times \overline{z-b}}{r} t = \text{Area}$.

Cor. As Radius :

Cotan. half the vertical Angle ::

So Rect. Base + diff. Sides, and Base — diff. Sides:
Four Times the Area of the Triangle.

For if $a = \text{S. } \frac{1}{2} \text{ Angle F}$, then by Sch. Pr. II. B. I.

$$s = \frac{2aa}{t} = \frac{2aa\tau}{rr}, \text{ and the Area } = \frac{mns}{2r} = \frac{mna^2\tau}{r^3}.$$

But by Cor. 3. Prop. VII. $mna^2 = z - m \times z - n \times rr =$
 $\frac{b+n-m}{2} \times \frac{b+m-n}{2} r^2$, therefore the Area $= \frac{b+n-m}{2}$
 $\times \frac{b+m-n}{2r} \tau.$

PROP.

P R O P. XVI.

If three Lines be drawn from the three Angles of a Triangle to any Point O, the product of the Sines of all the alternate Angles will be equal. 9.

Let the Sines of the Angles be as in the Figure, then in the Triangle

$$\left. \begin{array}{l} \text{POI, } a : f :: \text{IO} : \text{PO} \\ \text{IOH, } e : d :: \text{HO} : \text{IO} \\ \text{HOP, } c : b :: \text{PO} : \text{HO} \end{array} \right\} \text{these multiply'd}$$

$$aec : fdb :: \text{IO} \times \text{HO} \times \text{PO} : \text{PO} \times \text{IO} \times \text{HO}.$$

but the two last Terms are equal, and therefore the two first, $aec = fdb$.

Cor. 1. If two Angles, P, H, are bisected, by the Lines P O, H O; then the third Angle I will also be bisected. For since $ace = bdf$, and $a = b$, and $c = d$, or $ac = bd$, therefore $e = f$; and the contrary.

Cor. 2. Hence also three Perpendiculars erected on the middle of the three Sides of a Triangle, intersect in one Point. And the contrary.

For if two Perpendiculars be erected on two Sides PH, IH, then will $b = c$, and $d = e$, whence $a = f$, therefore the Perpendicular stands on the Middle of PI.

S C H O L I U M.

If the Point O, be taken out of the Triangle, this Prop. will hold equally true.

P R O P. XVII.

If four right Lines be drawn from the four Angles of a Trapezium, to any Point in it as O; the Products of the Sines of the alternate Angles, will be equal to each other. 12.

FIG. In the Triangle

$$\begin{array}{rclclcl}
 12. & \text{SPO, } a & : & b & :: & \text{SO} & : & \text{PO} \\
 & \text{POH, } c & : & b & :: & \text{PO} & : & \text{HO} \\
 & \text{HOI, } e & : & d & :: & \text{HO} & : & \text{IO} \\
 & \text{IOS, } g & : & f & :: & \text{IO} & : & \text{SO}
 \end{array}$$

their Product

$$aceg : bddf :: \text{SO.PO.HO.IO} : \text{PO.HO.IO.SO.}$$

therefore

$$aceg = bddf.$$

Cor. 1. After the same Manner, if Lines be drawn from all the Angles of any Polligon whatever to a Point; the Products of the Sines of the alternate Angles will be equal.

13. Cor. 2. In any Trapezium, let a, b, c, d & c . be the Sines of the Angles, as in the Figure; then

$$\text{As } bbd + ace :$$

$$bbd - ace ::$$

$$\text{Tan. half the Sum of the Angles, } \frac{ISH + SIP}{2} ;$$

$$\text{Tan. half the Difference } \frac{ISH - SIP}{2} .$$

For since $aceg = bddf$, therefore $ace : bbd :: f : g$
 $:: \text{SO} : \text{IO}$, and by Composition and Division, bbd
 $+ ace : bbd - ace :: \text{IO} + \text{SO} : \text{IO} - \text{SO} ::$ (by Prop.

$$\text{VI.) Tan. } \frac{\text{ISO} + \text{SIO}}{2} : \text{Tan. } \frac{\text{ISO} - \text{SIO}}{2} .$$



SECT.

S E C T. II.

The Solution of all the Cases of plain Triangles.

From the foregoing Propositions, all the Cases of plain Triangles may be resolved. Every Triangle has 6 Parts, 3 Sides and 3 Angles; and any three Things being given (except the 3 Angles) the other 3 may be found; but if only 3 Angles are given, there may be found an infinite Number of Triangles, that will have these three Angles. I shall here give the Solution of right Angled-triangles 3 different ways, *viz.* arithmetically, logarithmically, and algebraically: and that of oblique Triangles, logarithmically and algebraically. For it is needless to spend time in solving these arithmetically; since any oblique Angled-triangle is divided into two right ones by a Perpendicular, and then they are resolved by the Cases of right Angled-triangles; this Method is true to 3 or 4 places of Figures, and is sufficient in common Cases. I have also omitted their Solution by Projection and instrumentally, as being only Approximations to the Truth; and they are besides so easy in themselves, to any body that can but handle a Pair of Compasses, as to need no particular Explication. I shall only observe in general, with Regard to *Gunter's Scale*, &c. That any simple Proportion may be wrought on it by this *general Rule*.

Extend the Compasses from the first Term to one of the Means, on its proper Line; that extent, set the same way, upon its proper Line, will reach from the other Mean to the fourth Term required; where Radius is the Sine of 90° , or Tan. 45° . If, in extending upon the Tangents, the Compasses reach beyond the End; set it so far back as it reaches over.

Now

Now according to the different Variety of things given and sought, the Solution of Triangles is divided into several Cases as follows; in which I have only put the easiest Solutions. They that require more may consult the Propositions, from whence they are derived. I have added no numerical Examples, because any body can add and subtract, according to *this Rule*.

Add the Logarithms of the second and the third Terms together, from which subtract the Logarithm of the first Term, and the remainder is the Logarithm of the fourth Term sought, where Radius is 10.

But in algebraic Solutions you must use natural Sines, Tangents &c. and they must be actually multiply'd, and divided, and here Radius is 1.

Note, In right Angled-triangles if one acute Angle is given, the other is found by subtracting this from 90.

And in oblique Triangles, if 2 Angles be given, the third is found by subtracting their Sum from 180. or subtracting one from 180 gives the Sum of the other two.

In solving any of the Cases, it will be proper for distinction's sake to mark what is given with a Dash (/), and what is sought with a Cypher (o).



Right

Right angled plain Triangles.

C A S E I.

An Angle C and the Hypothenuſe BC, being given; to find a Side, BA. FIG. 14.

I. Arithmetically without Tables.

Let D = Degrees in the leſſer Angle; then

$$\frac{57.3}{D} + \frac{3D}{1000} : 1 :: \text{Hyp. BC} : \text{leſſer Side CA},$$

$$\text{or } 1 : 1 - \frac{1\frac{1}{2}DD}{10000} :: \text{Hyp. BC} : \text{greater Side BA}.$$

II. Logarithmically, by the Table of artificial Sines &c.

Rad : Hyp. BC :: S. an Angle C : op. Side BA.

III. Algebraically, by the Table of nat. Sines &c.

Let Hyp. BC = b , nat. Sine of C = s , Rad = 1. then

$$1 : b :: s : sb = \text{BA}.$$

C A S E II.

Given an Angle B or C, and a Leg; to find the Hypothenuſe, BC. 14.

I. Arithmetically.

Let D = Degrees in the leſſer Angle, then

$$1 : \frac{57.3}{D} + \frac{3D}{1000} :: \text{leſſer Side AC} : \text{Hyp. BC}.$$

$$\text{or } 1 - \frac{1\frac{1}{2}DD}{10000} : 1 :: \text{greater Leg BA} : \text{Hyp. BC}.$$

II. Logarithmically.

S. Angle B : opposite Side CA :: Rad : Hyp. BC.

III. Alge-

FIG.

III. Algebraically.

14.

Let $s = S.B$, $p = A.C$, then $\frac{p}{s} = \text{Hyp. } B.C.$

C A S E III.

14.

Given an Angle, and a Leg CA ; to find the other Leg, BA .

I. Arithmetically.

Find the Hyp. BC , by Case II. then $\sqrt{BC^2 - CA^2} = BA$.

II. Logarithmically.

Rad : one Leg $CA :: \text{Tan. Angle } C : \text{op. Side } BA$.

III. Algebraically.

Let $S.C = s$, $S.B = c$, $\text{Tan. } C = t$, $CA = p$, then

$$pt = BA, \text{ or } \frac{ps}{c} = BA.$$

C A S E IV.

14.

Given the Hyp. BC , and a Leg CA ; to find an Angle B , or C .

I. Arithmetically.

Find the other Leg, BA , by Case V. then

As Hyp. $BC + \frac{1}{2}$ the longer Leg $BA : \text{shorter Leg } CA :: 86 : \text{op. Angle } B$, and B taken from 90 gives C .

II. Logarithmically.

As Hyp. $BC : \text{Rad} :: \text{given Leg } CA : S. \text{op. Angle } B$, and B taken from 90 gives C .

III. Algebraically.

Let Hyp. $BC = b$, Leg $CA = p$, then $\frac{p}{b} = S.B$, and
 $90 - B = C$.

C A S E

CASE V.

FIG.

Given the Hypothenufe BC, and a Leg CA, to find the other Leg BA.

14.

I. Arithmetically.

$$\sqrt{BC^2 - CA^2} = BA.$$

II. Logarithmically.

Take the Sum and Difference of the Hyp. BC, and given Leg AC, add their Logarithms together; half the Sum is the Logarithm of the Side required, AB.

III. Algebraically.

Let $BC=b$, $CA=p$, then $\sqrt{bb - pp} = BA$.

Or let $BC+AC=z$, $BC-AC=d$, then $\sqrt{dz} = BA$.

CASE VI.

The Legs AB, AC, being given to find an Angle, B.

14.

I. Arithmetically.

Find the Hyp. BC by Case VII. then as Hyp. BC + half the longer Leg, BA : lesser Leg CA :: 86 : Angle op. B.

II. Logarithmically.

As one Leg AB : Rad :: other Leg AC : Tan. B, the op. Angle.

III. Algebraically.

Let $AB=b$, $AC=p$, then $\frac{p}{b} = \text{Tan. B}$ or $\frac{p}{\sqrt{bb+pp}} = \text{S.B.}$

CASE

FIG.

14.

C A S E VII.

The Legs AB , AC being given ; to find the Hypothenuſe BC .

I. Arithmetically.

$$\sqrt{AB^2 + AC^2} = BC.$$

II. Logarithmically.

Find the Angle B by Caſe VI. and then the Hyp. BC by Caſe II.

III. Algebraically.

Let $AB=b$, $AC=p$, then $\sqrt{bb+pp} = BC.$

*Oblique*

Oblique plain Triangles.

C A S E I.

Given two Angles A, B, and an opposite Side, FB; 15.
to find the other opposite Side A F. 16.

I. Logarithmically. 17.

S. one Angle A : op. Side F B :: S. the other
Angle B : to its opposite Side A F.

II. Algebraically.

Let $a = S.A$, $s = S.B$, $FB = d$ then $\frac{s d}{a} = A F$.

C A S E II.

Two Sides A F, F B, and an opposite Angle B 15.
being given ; to find the other opposite Angle A. 16.

I. Logarithmically. 17.

One Side A F : S. its op. Angle B :: other Side
F B : S. of its op. Angle A.

If the given Angle B is obtuse, or if its opposite
Side A F be greater than the other Side B F, then
Angle A sought is acute ; otherwise 'tis doubtful:
And the Sum of A and B taken from 180 gives F.

II. Algebraically.

Let $s = S.B$, $A F = c$, $F B = d$, then $\frac{s d}{c} = S.A$.

C A S E

FIG.

CASE III.

15.
16. Two Sides AF, FB, and an opposite Angle B,
17. given; to find the third Side AB.

I. Logarithmically.

Find the Angle A by Case the second, and then the Angle F will be had. Then find AB by Case I.

II. Algebraically.

Let $AF=c$, $BF=d$, $S.B=s$, $\text{Cof. } B=m$. Then $dm + \sqrt{cc - ssdd} = AB$; + if $\angle A$ is acute, and — if obtuse. Where note, if B be obtuse m will be negative

CASE IV.

15. Two Sides AF, FB, and the included Angle F,
16. being given; to find the other Angles, A, B.

17. I. Logarithmically.

As Sum of the Sides $AF+FB$: their diff. $AF-FB$:: Tan. half the Sum of the op. Angles $\frac{B+A}{2}$: Tan. of half their diff. $\frac{B-A}{2}$; then $\frac{B+A}{2} + \frac{B-A}{2} = B$ the greater Angle, and $\frac{B+A}{2} - \frac{B-A}{2} = A$ the lesser Angle.

II. Algebraically.

Let $AF=c$, $BF=d$, $x=S.F$, $y=\text{Cof. } F$. Then $\frac{dx}{c-dy} = \text{Tan. } A$, and $\frac{cx}{d-cy} = \text{Tan. } B$.

Or $\frac{c-dy}{\sqrt{dd+cc-2dcy}} = \text{Cof. } A$, and $\frac{d-cy}{\sqrt{dd+cc-2dcy}} = \text{Cof. } B$.

CASE

C A S E V.

Given two Sides AF, FB, and the contain'd Angle F; to find the third Side AB.

15.

16.

17.

I. Logarithmically.

Find A or B by Case IV. then AB by Case I.

II. Algebraically.

Let $AF=c$, $BF=d$, $y = \text{Cof. F.}$ Then
 $\sqrt{cc+dd-2cdy} = AB.$

See also the Corollaries to Prop. VII.

C A S E VI.

Three Sides given, to find an Angle F.

18.

I. Logarithmically.

Let fall a Perpendicular BD, on a Side adjoining to the Angle sought, then as Base AF : Sum. Sides AB+BF :: their diff. AB-BF : AD-DF the diff. Segments, then $\frac{1}{2}$ Base AF + $\frac{1}{2}$ diff. = greater Segment, and $\frac{1}{2}$ Base AF - $\frac{1}{2}$ diff. = lesser Segment. Then there is given BF, DF to find F by Case IV. of right angled Triangles. Also see Prop. VII. and Cor.

II. Algebraically.

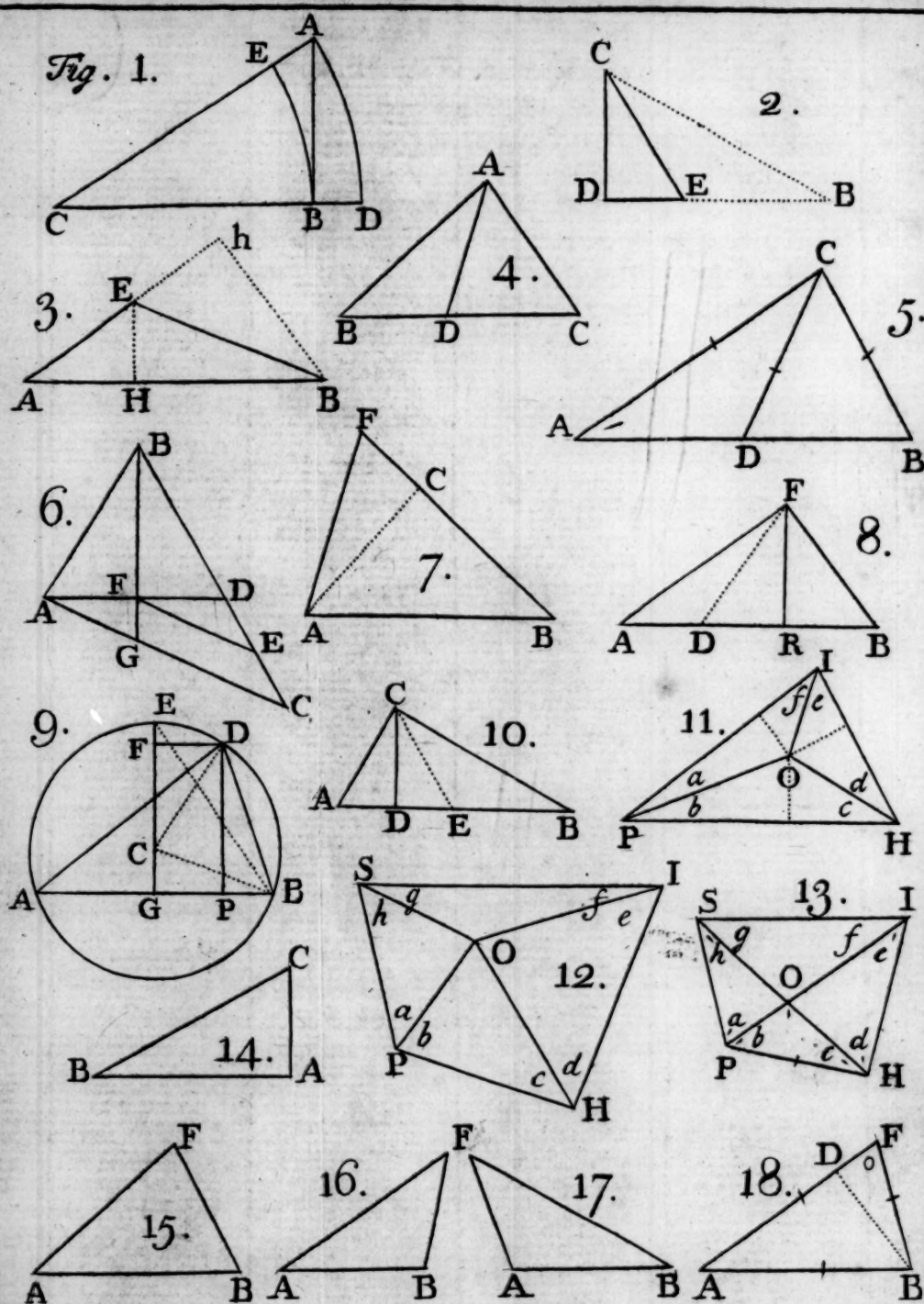
Let $AF=c$, $BF=d$, $AB=b$, $\left\{ \begin{array}{l} \text{then } \frac{cc+dd-bb}{2cd} = \text{Cof. F.} \\ \text{or } \frac{b^2-n^2}{2dc} = \text{vers. F.} \end{array} \right.$
 $AF \propto BF = n,$

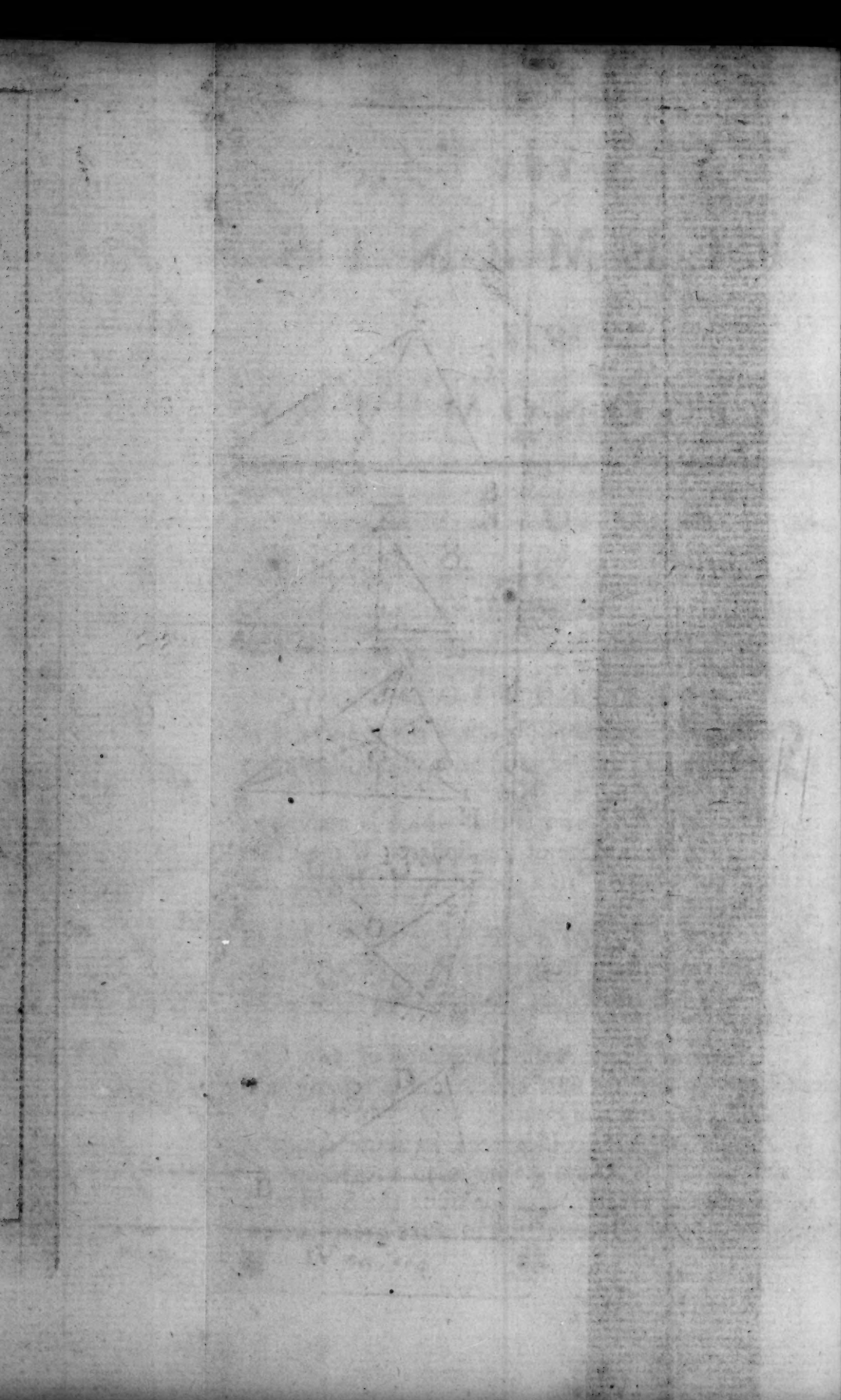
See also the Corollaries to Prop. VII.

SCHOLIUM.

In the Solutions of Plane Triangles there are three Cases, where the Thing sought requires two Operations, 1. When two Legs of a right angled Triangle are given to find the Hypothenufe, 2. When two Sides and the included Angle, in an oblique Triangle, are given, to find the third Side, 3. When two Sides and an Angle opposite are given, in an oblique Triangle, to find the third Side.







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THE
ELEMENTS
OF
TRIGONOMETRY.

B O O K III.

The Doctrine of the Sphere and spherical Trigonometry.

DEFINITIONS.

1. **S***pherical Trigonometry*, teaches the Properties of spherical Triangles, and how to calculate their Sides and Angles.
2. *A Circle of the Sphere* is that which is made by a Plane cutting the Surface of the Sphere. If the Plane pass thro' the Center, 'tis a great Circle : If not, it is a lesser Circle.
3. *The Pole of a Circle* is a Point on the Surface of the Sphere equidistant from every Point of the Circle. Every Circle has two Poles, diametrically opposite to each other.
4. *A spherical Angle* is the Inclination of two Circles of the Sphere to one another, intersecting in a Point call'd the angular Point.
5. *A right Angle* is 90 Degrees ; an acute Angle is less, and an obtuse Angle greater than a right one.
6. *A spherical Triangle* is made upon the Surface of the Sphere, by the Intersection of three great Circles.

If one Angle be right, 'tis called a *right angled* spherical Triangle. If one Side be a Quadrant 'tis called a *quadrantal* Triangle. If no Angle or Side be 90° , 'tis an oblique Triangle.

7. In a right angled Triangle, the Side opposite to the right Angle is called the *Hypotenuse*; the other two are called *Legs* or Sides.

8. In a right angled or quadrantal Triangle, the five circular Parts (setting aside the right Angle or quadrantal Side) are, the two Parts adjoining on each Side the right Angle or quadrantal Side, and the Complements of the three Parts which are furthest distant from it. The *middle Part* is that which is equidistant from other two; which two if they adjoin to the middle Part are called *extreams conjunct*; and if they be remote from it, are called *extreams disjunct*.

9. Sides and Angles are said to be of the *same Affection*, when both are greater, or both lesser, or both equal to 90° . And they are of *different Affection*, if one is greater and another less than 90° .

10. *Like or similar Arches* of different Circles are those that contain the same Number of Degrees.



SECT.

S E C T. I.

The Properties of spherical Angles and Arches.

P R O P. I.

The Section of a Sphere and a Plane is a Circle.

Let BIC be the Section, and let the Plane of the great Circle ABCD be drawn thro' the Center E of the Sphere, perpendicular to the Plane BIC, and let BHC be their common Interfection. Draw EH perpendicular to BC, then $BH=HC$. Take any Point I in it, and draw HI, IC; and also BE, CE. Since EH is perpendicular to BC, and in the perpendicular Plane ABCD, therefore 'tis perpendicular to the Plane BIC, and therefore to HI. But BE, CE, IE are equal Radii of the Sphere; therefore in the right angled Triangles BEH, IEH; $BH^2 + HE^2 = BE^2 = IE^2 = HI^2 + HE^2$, therefore $BH=HI$, and $BH=HI$, therefore BIC is a Circle whose Center is H.

FIG.
19.

Cor. 1. If a great Circle ABCD be perpendicular to any other Circle BIC it passes thro' its Poles.

For since it is prov'd that H is the Center of the Circle BIC, and EH perpendicular to it, produce EH to P which will be in the Plane of the Circle ABD, because EH is in it. Draw the great Circle PJ, then since the Sines BH, HI, HC are equal, therefore the Arches belonging to them BP, IP, CP are equal, therefore P is the Pole. And if PE was produced to the other Pole, that Pole will also be in the Plane of the Circle ABC.

Cor. 2. If a great Circle ABC passes thro' the Pole P of another Circle, it cuts it at right Angles and into two equal Parts.

FIG. For draw P E to cut the Circle in H, then is H
 19. the Center, and P H perpendicular to its Plane, and
 consequently E H and the Plane A B C D are perpendi-
 cular to it. And since the Center H is in B C, the
 common Intersection of the Planes; therefore B I C is
 a Semicircle.

P R O P. II.

20. *All great Circles of the Sphere cut one another into
 two equal Parts.*

For the common Section of their Planes is the Dia-
 meter of the Sphere, and consequently their Segments
 are Semicircles.

Cor. 1. Any Side of a spherical Triangle is less than
 a Semicircle.

For since A D C or A B C is a Semicircle, therefore
 in the Triangle A D B, A D or A B is less than a Se-
 micircle.

Cor. 2. If two Sides of a spherical Triangle A B D
 be produced, till they intersect in C, each will become
 a Semicircle.

P R O P. III.

21. *If one or more Circles intersect another Circle in one
 and the same Point C; the Sum of the Angles A C E,
 E C D is equal to two right Angles.*

Let the Arch E C intersect A D in C, and draw
 C B perpendicular to A D; then $ACB + BCD =$ two
 right Angles, but $ACE + ECB = ACB$, and ECD
 $- ECB = BCD$, therefore $ACE + ECD = ACB$
 $+ BCD =$ two right Angles.

Cor. 1. All the Angles made about one Point are
 $=$ four right Angles.

Cor. 2. When one Circle crosses another, the ver-
 tical Angles a, c , are equal.

For

For $a+b$ = two right Angles, and $b+c$ = two right FIG. Angles, therefore $a+b=b+c$ and $a=c$.

P R O P. IV.

A spherical Angle made by two great Circles is measured by the Arch of a great Circle intercepted between the Sides, and described at a Quadrant's distance from the Angular Point. 22.

Let $AB=AD=90^\circ$, and draw the Arch BD , and from the Center E draw BE , DE ; also draw the Tangents AT , AS , to the Arches AB , AD ; then the $\angle TAS$ = spherical Angle BAD , and AEB , AED will be two right Angles, but TAE , SAE are right Angles; therefore TA , SA are $\parallel$ to BE , DE , and $\angle TAS=BED$. But the Arch BD is the Measure of the Angle BED or TAS , and therefore of the spherical Angle BAD .

Cor. 1. The Angles BAD , BCD , made at opposite Points of the Semicircle are equal. For the $\angle BED$ or Arch BD is the Measure of both.

Cor. 2. The Distance of the Poles of two Circles is equal to the Angle of their Inclination.

For since AE is $\perp$ to the Plane BDE , this Plane is $\perp$ to both the Planes ABC , ADC ; therefore the Poles are in the Circle BD , as suppose at P , Q . Then $BP=90^\circ=DQ$, and subtracting DP from both, then $BD=PQ$.

Cor. 3. Two great Circles ABC , ADC passing thro' the Poles of another great Circle BD , will cut all the Parallels to BD , as bd , into similar Arches.

For AC is $\perp$ to bed , and e the Center of bd , therefore $\angle BED=bed$, or the Arch bd similar to BD .

Cor. 4. Hence an Angle made by two great Circles of the Sphere, is equal to the Angle of Inclination of the Planes of these great Circles. For $\angle BED$ or $bed=\angle TAS$ or BAD .

FIG. *Cor. 5.* Hence also if bd be a Parallel to the great Circle BDP ; it is as Radius: Cosine of the Parallel's distance from its great Circle :: so any Arch of the great Circle: to a similar Arch of the Parallel.

For $BE:BD::be:bd$.

P R O P. V.

23. *If any two Circles AEB, AFB, intersect one another on the Sphere, they make the opposite Angles A and B equal, and the opposite Parts AFE, BFE similar.*

Let AB be the common Section of their Planes; and in the Planes of their respective Circles, draw the Tangents AC , BC to the Circle AFB ; and AD , BD to the Circle AEB , and join CD . Then in the Triangles ACD , BCD , $AC=BC$, being Tangents to the same Circle and for the same Reason $AD=BD$, and CD is common; therefore the Triangles are equal and similar, and the $\angle CAD= CBD$, and therefore the Spherical Angles A and B equal to them, are also equal to one another.

Likewise the opposite Parts of the Figure $AFBEA$, are similar, being bounded by Parts of the same Circles, having the same Position.

Cor. Hence any two like Arches OI , SV , drawn after a like manner, will be equal. That is if $AO=BS$, and $AI=BV$, or $\angle AOI=\angle BSV$, then $OI=SV$. For then the Triangles AOI , and BSV are similar and equal.

S C H O L I U M.

Tho' the opposite Parts of the Figure AFE and BFE are said to be similar; yet if the Angle A be apply'd to B , the Triangles AFE , BFE will not coincide. Or if the Base FE of one be laid upon the Base EF of the other, the triangular Figures AFE , BFE will lye contrary ways, one to the right, the other to the left hand.

P R O P.

P R O P. VI.

24.

If two great Circles HO, HR have the same Inclination to a third EQ; the Arches of all the Parallels (AP) to the third, intercepted by the other two, will be similar Arches.

Let the great Circle EC, cut the Circles HO, HR at equal Angles in E, and Q. Thro' the Poles of EC draw the great Circles PB, AC. Then in the Triangles PEB, AQC, $\angle E = \angle Q$, and the Angles at B and C are right, and $PB = AC$: therefore if the Angle E be laid upon the Angle Q, then will P fall somewhere in the Arch PA (because $PB = AC$); and somewhere in the Arch QH (because $\angle E = \angle Q$), therefore P falls upon A, their Point of Intersection; and consequently B upon C (otherwise A would be the Pole of CQ); therefore $BE = CQ$, and $PE = AQ$, therefore $EQ = BC$; but BC is similar to AP (by Cor. 3. Prop. IV), therefore EQ and AP are similar.

Cor. The same Things supposed, the Arches of the two inclined great Circles intercepted between any two Parallels to the third, are equal, $PE = AQ$.

P R O P. VII.

25.

If a Plane EPQ. be supposed to be drawn perpendicular to the Planes of two equal Circles HO, EQ; and any other Plane PDKN be drawn thro' the two Poles P, N, which are equidistant from their respective Circles, and to revolve about PN: Then these Planes will intercept equal Arches of these Circles, DO, KQ.

For let N be the Pole of HO, and P of EQ, and $NO = PQ$, then by Cor. Prop. V. $\angle NOH$ or $NOD = PQE$ or PQK ; and $\angle POD = NQK$, and $NQ = PO$, and by Prop. V. $\angle N = P$. Therefore the Triangles POD and NQK are similar and equal, and

FIG. and therefore Arch $DO = KQ$. And likewise Arch $HD = EK$.

P R O P. VIII.

26. *Of several Arches of great Circles drawn from the same Point A to any other Circle CED; the greatest AD is that which passes thro' the Pole P, and the nearer to this as AB is greater than that which is further off as AE. And the least is AC the remainder of that which passes thro' the Pole.*

Let AS be perpendicular to the Plane of the Circle CED, O its Center, CSOD its Diameter. Draw the right Lines SB, SE, AD, AB, AE, AC. Then you have the right angled plane Triangles ASD, ASB, ASE, ASC, wherein the Perpendicular AS is common. But (Eucl. III. 7.) the Base SD passing thro' the Center is the greatest, and SB greater than SE, and SE greater than SC which is the least.

Therefore the Hypothenuse AD is the greatest, AC the least, and AB greater than AE. But these are the Cords of Arches, and to greater Cords belong greater Arches, consequently the Arch AD is the greatest, Arch AB (nearer to AD) is greater than Arch AE (further from it); and Arch AC the least of all.

27. *Cor. 1. If two Sides in two spherical Triangles APE, APB, be respectively equal; that which hath the greater included Angle hath the greater Base, and the contrary.*

For about the Pole P, at the Distance PE or PB (the greater side in the two Triangles), describe the Circle EBD, and from A thro' P draw the great Circle APD. Then by this Prop. if BD be less than ED, or $\angle APB$ greater than APE; then is AB greater than AE.

And if AB be greater than AE, then AB is nearer to D than AE, $\angle APB$ greater than APE.

Cor. 2. The Perpendicular let fall on the Base of a spherical Triangle, is either greater than either Side, or less than either Side. For

For it passethro' the Pole of the Base, and is therefore either the longest or shortest Line drawn from the Vertex of the Triangle.

Cor. 3. And therefore when the Perpendicular falls without the Triangle, either the greater or lesser Perpendicular may be esteem'd the Perpendicular upon the Base.

Cor. 4. Hence if the Perpendicular fall without, as in the Triangle B A E, the greater Perpendicular A D lyes next the greatest Side A B; and the least Perpendicular A C next the least Side A E.

28.

But when the Perpendicular falls within; if it is less than a Quadrant as A C (in the Triangle A E F), then it lyes nearer the lesser Side A E, and the Segment C E is less than C F, and $\angle C A E$ less than $C A F$. But if it is greater than a Quadrant, as A D (in the Triangle A B G), then it lyes nearest the greater Side A G, and then it makes the lesser Segment of the Base (D G), and lesser vertical Angle (D A G), next the greater Side A G. And the greater ones next the lesser Side A B.

Cor. 5. The Perpendicular falling within, or the nearest Perpendicular falling without, is of the same Affection, as half the Sum of the Sides.

For in the Triangle E A F, A C is less than a Quadrant, and since $A F < A G$, therefore $A E + A F < E G$ or 180 , and $\frac{A E + A F}{2} < 90$. But in the Triangle G A B, A D is $>$ a Quadrant; and since $A G > A F$, therefore $B A + A G > B F$ or 180 , and $\frac{B A + A G}{2} > 90$.

Again in the Triangle B A E, where the Perpendicular A C is less than a Quadrant, and $C E < C F$; then by this Prop. $A E < A F$, and $A E + A B < B F$ or 180° , and $\frac{A E + A B}{2} < 90$. But in the Triangle G A F, where the Perpendicular A D (greater than

FIG. than a Quadrant) is nearest, or $DB > DG$, or CF
 28. $> CE$, then $AF > AE$, and $AF + AG > GE$ or
 180. and $\frac{AF + AG}{2} > 90$.

Cor. 6. Hence when the Perpendicular falls within, or you take the nearest Perpendicular falling without; then if half the Sum of the Sides be less than a Quadrant, the lesser Segment, and lesser vertical Angle adjoins to the lesser Side; but if half the Sum of the Sides be greater than a Quadrant; the lesser Segment and lesser vertical Angle adjoins to the greater Side.



SECT.

S E C T. II.

The Affections of spherical Triangles.

P R O P. IX.

FIG.

Every Triangle ABC hath (by producing its Sides) 29.
 another Triangle abc , on the opposite Side of the Globe,
 similar and equal to it.

For the Point a is opposite to A , or a Semicircle distant from it; and b is opposite to B , and c to C . Therefore $ab = AB$, $bc = BC$, and, $ac = AC$. Likewise by Cor. 1. Prop. IV. $\angle a = \angle A$, $b = B$, and $c = C$; therefore the Triangle abc is equal and similar to ABC .

Cor. I. Hence any Triangle ABC , by producing its Sides, hath opposite to every Angle thereof, another Triangle of the same Base and opposite Angle with the former; and the other Parts the Supplements thereof.

Thus in the Triangle Bca , $\angle B = \angle ABC$, $ac = AC$, $aB = \text{Sup. } AB$, $cB = \text{Sup. } CB$, $\angle caB = \text{Sup. } CAB$ or A , and $\angle acB = \text{Sup. } C$.

Cor. 2. Any Triangle ABC , has adjoining to every Side thereof another Triangle having the same Side and opposite Angle, and the other Parts the Supplements thereof.

Thus in the Triangle ABc , AB and $\angle c$ is the same with AB and $\angle C$; Bc , Ac are the Supplements of BC , AC ; $\angle cBA = \text{Sup. } CBA$, and $\angle cAB = \text{Sup. } CAB$. Likewise there is the Triangle aBC adjoining to CB ; and the Triangle AbC adjoining to AC .

SCHO-

FIG.

29.

SCHOLIUM.

Tho' the Triangles ABC , abc are said in this Prop. to be similar; yet they will not coincide when apply'd to one another. For laying any Angle a upon its equal A , so that the Concavities of the Triangles lye the same way; the Side ac will fall on AB , and ab on AC .

PROP. X.

30.

If from the three Angles of any spherical Triangle ABC , as Poles, you describe three great Circles, they will form another spherical Triangle DEF by their Intersections; each Side and Angle whereof will be the Supplements of the Angle and Side opposite, in the given Triangle.

Produce the Sides as in the Fig. then since Bk or BF is a Quadrant, and Ag or AF is a Quadrant, therefore F is the Pole of AB ; and likewise E the Pole of AC , and D the Pole of CB . then $DQ = a$ Quadrant $= ER$, and $DE = QR = \text{Sup. } \angle C$.

Also $Eg = 90^\circ = Fp = Tp$. and $TE = pg$, therefore $FE = \text{Sup. } ET$ or pg that is of the $\angle CAB$. Likewise $DT = kr$ or $\angle B$, and $DF = \text{Sup. } \angle B$.

Also $rB = 90^\circ = Ap$, and $rB + Ap$ or $rp + AB = 180^\circ$, and rp or the $\angle F = 180 - AB = \text{Sup. } AB$.

And thus $Ag = CR$, and $AC = gR$, and $\text{Sup. } gR = \text{Sup. } AC$, that is $\angle E = \text{Sup. } AC$.

Lastly $CQ = 90^\circ = Bk$, and $CQ + Bk$ or $kQ + CB = 180$, and $kQ = 180 - CB$, that is $\angle D = \text{Sup. } CB$.

Cor. 1. Some three Poles of the Sides of any Triangle, form another Triangle, wherein the Angles and Sides are respectively equal to the Supplements of the Sides and Angles of the other; and these two Triangles are mutually supplemental to each other.

For D, E, F are the Poles of the Sides of the Triangle

angle ABC , as well as A, B, C are the Poles of the FIG. Sides of DEF .

30.

Cor. 2. The nearest Poles D, E, T of the Sides of any spherical Triangle ABC , form another Triangle, wherein the Angles and Sides are respectively equal to the Sides and Angles of the given one: Excepting only the Supplements of one Side and its opposite Angle in the former, will be the correspondent Angle and Side in the latter.

For in the Triangles DFE, DTE , DE is common, and $\angle F = \angle T$; and all the Rest are the Supplements. There are other five, or in all six such Triangles, and all adjoining to the Angles and Sides of the Triangle DEF , by *Cor. 1, 2, Prop. IX.* but there is only one supplemental Triangle DEF to the Triangle ABC .

SCHOLIUM.

By this *Prop.* quadrantal Triangles may be reduced to right angled ones.

PROP. XI.

In two Triangles, if three Sides in one be respectively equal to three Sides in the other; or the three Angles in the one, to the three Angles in the other; These Triangles will be equal in all respects.

Case 1. If three Sides be equal; and if the Base of one Triangle be laid on the Base of the other, the two other Sides must coincide; because there can be but one Point from whence two given Arches can be drawn to the two Angles at the Base.

Case 2. If three Angles are equal, then by *Cor. 1. Prop. IX.* the Sides of two Triangles form'd by the Poles of both, will be the Supplements of the Angles of the former, and therefore respectively equal. Therefore by *Case 1* the Angles of these supplemental Triangles are equal, and these Angles are the Supplements of the Sides of the former Triangles by *Cor. 1. Prop.*

FIG. Prop. X. therefore the Sides of the former Triangles are respectively equal.

SCHOLIUM.

This Demonstration supposes the equal Sides to lye the same way ; but if they lye contrary ways, they cannot concide. Yet if you will suppose the Convexity of one of them to lie the contrary way, then they will coincide when laid upon one another. The same may be apply'd to the following Prop.

PROP. XII.

If in two Triangles, there be two Sides and the included Angle ; or two Angles and the included Side, respectively equal ; the two Triangles will be equal in all respects. For,

1. If one side in one Triangle be laid upon its Equal in the other Triangle ; then by reason of the included Angle being equal in both ; the other two Sides will coincide, and so the whole Triangle.

2. If the equal Side in one be laid upon that in the other, and the equal Angles upon one another, the whole Triangles will coincide, and will therefore be equal.

PROP. XIII.

The Sum of any two Sides of a spherical Triangle is greater than the third Side.

31. Let AB be the greatest Side ; about the Pole B, thro C describe the Arch CD, which will be $\perp$ to BD, also about the Pole A thro' D describe the Arch DE, which will also be $\perp$ to AD. And since the two Circles CD, ED, can but touch in one point D, therefore at E, C there is a space EC between them. Whence, since $AE + CB = AD + DB$ or AB, therefore $AE + EC + CB$ or $AC + CB$ is greater than AB.

Cor.

SECT. II. of TRIGONOMETRY.

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Cor. 1. The Sum of the three Sides of any spherical Triangle DFE is less than a Circle. FIG.

For DE is less than DT+TE, and therefore DE +EF+FD is less than FDT+FET or two Semi-circles. 30.

Cor. 2. A great Circle is the nearest Distance of any two Points upon a Sphere. For it is a Line of the least Curvature that can be drawn, or deflects least from a right Line.

P R O P. XIV.

In any Triangle equal Sides are opposite to equal Angles. 32.

1. Let $AC=BC$, let CD bisect the Angle ACB, then in the Triangles ACD, CDB, there are two Sides and the included Angle at C respectively equal; therefore by Prop XII, the $\angle A=\angle B$.

2. If $\angle A=\angle B$, make $AE=BE$, then by Case 1. the $\angle EAB=\angle B=\angle CAB$ by Supposition; therefore E falls upon C, and $AC=BC$.

Cor. 1. An equilateral Triangle is also equi-angular, and the contrary.

Cor. 2. In an Isosceles Triangle, a Perpendicular let fall on the Base, bisects it and the vertical Angle: *et contra.*

Suppose the Convexity of the Triangle CDB to lye the contrary way; then because the Angles at D are equal, if CDB be laid upon CDA, DB will fall upon DA; and because $CB=CA$, and $\angle A=\angle B$ (by this Prop.), therefore the Point B will fall upon A; and the Triangles will coincide, then will $AD=DB$, and $\angle$'s at C will be equal.

Again, If $AD=DB$, or $\angle ACD=\angle BCD$; then by Prop. XII. the Angles at D are equal, or $CD\perp AB$.

K

P R O P.

FIG.

P R O P. XV.

31. *In any spherical Triangle, the greatest Side is opposed to the greatest Angle; and the least to the least.*

1. Let $\angle C > B$, and make $\angle BCD = B$. then will $CD = BD$ by Prop, XIV. but $CD + AD$ or $AB > AC$ by Prop. XIII. For the same Reason if $B > A$, then $AC > CB$.

2. Let $AB > AC$, then Angle $C > B$; for if C be equal or less than B , then AB is equal or less than CA by Case 1, which is contrary to the Supposition.

P R O P. XVI.

33. *If one Side DA of a Triangle DAB be produced, and if the Sum of the other sides $DB + BA$ be greater, equal, or less than a Semicircle; the internal Angle at the Base D, is accordingly greater, equal or less than the outward opposite Angle BAC.*

For if $DB + BA$ be $>$ or $<$ than DC , then BA is $>$ or $<$ than BC ; and by Prop. XV. the Angle C or D is $>$ or $<$ than BAC , respectively.

Cor. 1. In an Isocles Triangle, as one of the equal Sides is greater, equal, or less than a Quadrant; the Angle at the Base is accordingly greater, equal, or less than a right Angle.

Cor. 2. In an Isocles Triangle, either Side is of the same Affection as its opposite or adjacent Angle.

P R O P. XVII.

33. *In any spherical Triangle, if the sum of any two Sides be greater, equal, or less than a Semicircle; the Sum of the opposite Angles, is accordingly greater, equal or less than two right Angles.*

For

For by Prop. XVI, if $DB+BA$ be greater, equal or less than DC , then $\angle D$ is greater, equal, or less than BAC , and $D+DAB$ greater, equal, or less than $DAB+BAC$ or two right Angles. FIG. 33.

Cor. 1. In any Triangle half the Sum of any two Sides is of the same Affection as half the Sum of their opposite Angles.

Cor. 2. Hence it follows, that if in a Triangle ACB , there be given two Sides AC , CB , and an opposite Angle A ; and if CD be made $=CB$, then you will have two Triangles ACB , ACD containing the Things given, and therefore the other opposite Angle may be either B or ADC ; and consequently in both of them the Sum of the Sides AC , BC ; or AC , DC ; are of the same Affection (with respect to 180°), as the Sum of the opposite Angles A and B , or of A and ADC . Therefore whenever the Sum of the Angles A and B are of the same Affection as the Sum of A and the Supplement of B , then tis doubtful whether the other Angle B is acute or obtuse, because it may be either. But when these are of different Affections, that value only of B (whether acute or obtuse) must be taken, which when, added to A , their sum is of the same Affection, as the Sum of the opposite Sides. 34.

Cor. 3. And for the same Reason, when two Angles A and B , and an opposite side CB are given: If there be two Triangles ACB , DCB that contain the same *data*, it will be doubtful whether the other opposite Side be AC or its Supplement CD (for $CD=CE$, the Angles A, D, E being equal) for it may be either of them. Therefore we must always take that Value of AC which added to CB , their sum may be of the same Affection (with respect to 180°), as the Sum of the opposite Angles. And if both sums (*viz.* of AC or its Supplement, added to A) be of the same Affection; then it is doubtful. Also if CI be $\perp AB$, it will be ambiguous whether AI or its Sup. $EI=DI$ be the 35.

FIG. Segment of the Base, and whether A C I or its Sup. E C I or D C I be the Angle at the Vertex.

36.

Note, the Cases mentioned in the two last Corol. contain the six ambiguous Cases of oblique Triangles. For when two Sides and an Angle opposite are given; the other three Things will have different Values, according as an Arch or its Supplement is taken for the other opposite Angle. And if two Angles and an opposite Side be given; according as an Arch or its Supplement is taken for the other opposite Side; the Base and its opposite Angle will have different Values.

Cor. 4. If in a right angled Triangle, there be given an Angle and its opposite Side, there will be two Triangles D B A, C B A that have the same *Data*, and therefore the other three Things are doubtful. Whence the Hypothenuse may be either D B or its Sup. B C. If D B be the Hypothenuse, then the Base is D A and $\angle$ opposite D B A. If B C is the Hyp. then A C is the Base and A B C the op. Angle, these three being the Sup. of the others. And these are called the three ambiguous Cases in right angled Triangles; when an Angle and its op. Leg is given, to find the rest.

After the same manner in a quadrantal Triangle, if a Side and the opposite Angle be given; the Angle opposite to the quadrantal Side may be either that Angle D A B, or its Supplement B A C. And therefore the three things sought are ambiguous, and each of them may be either an Arch or its Supplement.

P R O P. XVIII.

In any Triangle A B C. the sum of the three Angles is greater than two right Angles.

30.

Let F D E be the supplemental Triangle to A B C, by Prop. X. then by Cor. 1. Prop. XIII. $D F + F E + E D$ is less than a Circle; that is the Supplements of A, B, and C is < 2 semicircles, or $180 - A + 180 - B +$

$B + 180 - C < 360$. subtract 2 semicircles, and then FIG.
 $180 - A - B - C < 0$ add $A + B + C$. then $180 < 30$.
 $A + B + C$, or $A + A + C > 180$.

Cor. 1. The sum of the three Angles of any Triangle is less than six right Angles. For every Angle must be less than 2 right Angles, otherwise it would be one continu'd Arch.

Cor. 2. In any Triangle D A B, the external Angle B A C is less than the Sum of the interior opposite Angles D and B. 33.

Cor. 3. The Sum of any two Angles is greater than the Supplement of the third Angle.

Cor. 4. The Sum of any two Angles—the third (or any Angle and the Difference of the other two) is less than two right Angles.

For F D is less than F E + E D, by Prop. XIII, that is $180 - B$, is less than $180 - A + 180 - C$, that is $180 + A + C - B$ is less than $180 + 180$, or $A + C - B$ less than 180 . 30.

Cor. 5. In a right angled spherical Triangle, the Sum of the oblique Angles is greater than one, and less than three right Angles. By this Prop. and Cor. 4.

P R O P. XIX.

If the Angles at the Base of a spherical Triangle, be of the same Affection, the perpendicular on the Base will fall within the Triangle; if of different Affection it falls without. 28.

In the Triangle G A B whose Base is G B and its Pole P, Since the Perpendicular must pass thro' the Pole P, by Prop. 1. Cor 1. let it intersect the Base in D, C, then the Angles P G B and P B G are right, and therefore A G B and A B G are both obtuse, when the Perpendicular C D falls between them. Likewise the Angles P F E, and P E F are both right; therefore in the Triangle F A E the Angles F and E are both

FIG. acute, when AC falls within. But in the Triangle
 28. BAE the Perpendicular falls without, and the $\angle ABE$ is acute and AEB obtuse. And the same will hold for any other Triangle that can be drawn.

36. *Cor.* If the two least Sides of a spherical Triangle, AC , BC be of the same Affection; the Perpendicular CD upon the other AB , will fall within the Triangle.

Let AB be the greatest Side, make $BE=BC$, and $AF=AC$, then by *Cor.* 2. *Prop.* XVI. the $\angle BEC$ is of the same Affection as CB . and AFC of the same Affection, as AC ; but AC , BC are of the same Affection, and therefore the Angles EFC , FEC . Therefore by this *Prop.* the Perpendicular CD falls within the Triangle ECF or ACB .

If AB be less than AC , CB , but greater than their Supplements (they being of the same Affection), the Perpendicular will fall within. For this and the adjoining Triangle (whose common Base is AB) have the same Perpendicular, see *Cor.* 2 *Prop.* IX.

SCHOLIUM

Some other Properties of Perpendiculars may be seen in the corollaries to *Prop.* 8.

PROP. XX.

37. *In any right angled spherical Triangle, each of the oblique Angles is of the same Affection as the opposite side.*

If the Angle at A be right, and AC less than a Quadrant, produce AC to D , that AD may be a Quadrant, and draw BD ; then D is the Pole of BA , and consequently ABD a right Angle, and therefore ABC is less than a right Angle. But if AC be greater than a Quadrant; then its opposite Angle ABE is greater than the right Angle ABD . And if AD be a Quadrant, the Angle ABD is a right Angle.

On

On the contrary, if ABD is a right Angle, then D is the Pole of AB , and AD a Quadrant; but if ABC be less or ABE greater than a right Angle; then the opposite Side AC will be less, or AE greater than the Quadrant AD . FIG. 37.

P R O P. XXI.

In a right angled spherical Triangle BAD , if the Legs be of the same Affection, the Hypothenufe BA , will be less than a Quadrant; if of different Affection, it will be greater: And the contrary. 38.

Let D be the right Angle, and produce DB , DA to C ; and make DP , DE Quadrants, then P is the Pole of DB , and E of DA , and PB a Quadrant. Then in the Triangle DBA , if DB , DA be less than a Quadrant, then BA will be less than the Quadrant BP , by Prop. VIII. And in the right angled Triangle BAC , if BC , AC , be greater than a Quadrant; then also BA is less than BP .

But in the Triangle DBF , where DB is less and DF greater than a Quadrant; as likewise in the Triangle BCF , the Hypothenufe BF will be greater than the Quadrant BP . by the same Prop. VIII.

On the contrary, if the Hypothenufe BA be less than a Quadrant BP , then DB , DA are both less; or else CB , CA both greater than a Quadrant. But in the Triangle DBF or CBF , where the Hypothenufe BF is greater than the Quadrant BP ; DB , DF or else CB , CF are of different Affections.

Cor. 1. As the two oblique Angles are of the same or different Affection; the Hypothenufe will be accordingly lesser or greater than a Quadrant: *et contra*: by Prop. 20, and 21.

Cor. 2. As the Hypothenufe and one Side (or its opposite Angle) is of the same or different Affection; the other Side (and its opposite Angle) will be less or greater than a Quadrant.

FIG. Cor. 3. As the Hypothenuſe is leſſer or greater than
 38. a Quadrant, each Leg will be ſimilar or diſſimilar
 to its adjacent Angle: and the contrary.

P R O P. XXII.

39. *If a Triangle C I A have one acute Angle, it alſo
 hath one Side leſs than Quadrant.*

Let A C I be the acute Angle, and in the Triangle
 C B A, let C be a right Angle; if either A fall be-
 tween P and C, or B between Q and C, the Prop.
 is manifeſt; for then either C A or C B is leſs than a
 Quadrant. Let therefore C B, C A be greater than
 Quadrants, then by Prop. 21. B A is leſs than a
 Quadrant, therefore in the acute angled Triangle
 A C I, A I is ſtill leſs than a Quadrant.

Cor. If a Triangle have one Side greater than a
 Quadrant, it alſo hath one obtuſe Angle.

For if a Triangle hath one Side greater than a Qua-
 drant, its ſupplemental Triangle (by Prop. X) hath
 one acute Angle; and therefore by this Prop. it hath
 a Side alſo leſs than a Quadrant. And if ſo, then (by
 Prop. X.) the former has an Angle greater than a
 right Angle.

P R O P. XXIII.

*If a Triangle has two Sides leſs than Quadrants, it
 has alſo an acute Angle.*

39. For in the Triangle D B A, if B D, D A be leſs
 than Quadrants, the Sum of the oppoſite Angles is
 leſs than 2 right Angles, by Prop. XVII. therefore
 at leaſt one of the Angles is leſs than a right Angle.

Cor. If a Triangle A C B has two obtuſe Angles,
 it has alſo one Side greater than a Quadrant.

For by Prop. XVII. If two Angles be greater
 than two right Angles, then the Sum of their op-
 poſite

posite Sides is greater than a Semicircle ; therefore at least one of them is greater than a Quadrant.

FIG.

P R O P. XXIV.

If the three Angles of a spherical Triangle be acute ; each Side will be less than a Quadrant. 36.

For the Perpendicular CD from any Angle will fall within the Triangle by Prop. XIX. and since, in the right angled Triangles ACD , and DCB , the Angles A and ACD , and B and BCD are all acute, therefore by Cor. 1. Prop. XXI. the Hypothenuses AC , BC are less than a Quadrant. And by the same Reasoning AB is less than a Quadrant.

Cor. 1. If the three Sides of a spherical Triangle DEF be greater than Quadrants, the three Angles will be obtuse. 30.

For then the three Angles A , C , B of the supplemental Triangle ACB , will be acute, and therefore the sides less than Quadrants, by this Prop. And therefore by Prop. X. the Angles of the Triangle DEF will be obtuse.

Cor. 2. In a Triangle DEF . if two Angles D , E be obtuse, and one F acute ; the Sides are of the same Affection as the opposite Angles.

For in the adjoining Triangle DET all the Angles are acute by Cor. 2. Prop. IX ; and all the Sides less than Quadrants by this Prop. and therefore in the Triangle DEF ; DF , EF are greater than Quadrants.

Cor. 3. In a Triangle, if two Sides be lesser, and one greater than a Quadrant ; the Angles will be of the same Affection as the opposite Sides. This follows from Cor. 2. of this, and Prop. X.

PROP. XXV.

39. *In a spherical Triangle, when two Sides are of the same Affection, and the included Angle acute; then the third Side is less than a Quadrant. But if the two Sides are of different Affection, and the included Angle obtuse; the third Side is greater than a Quadrant.*

1. In the Triangle BDA or BCA , right angled at D and C , if the Legs are of the same Affection, the Hypothenuse BA is less than a Quadrant, by Prop. XXII. but if the Angle at D or C be less than 90 , as ACI , ADI ; then AI is still less.

2. Suppose the Legs to be of different Affection, then by Prop. XXI. the Hyp. BA will be greater than a Quadrant, and much more will BA be greater than a Quadrant, if the Angle at C and D be obtuse,

Cor. When two Angles are of the same Affection, and the included side more than a Quadrant; then the third Angle is obtuse. But if two Angles are of different Affection, and the included Side less than a Quadrant, the third Angle is acute.

For by Prop. X. its supplemental Triangle comes under the Case of this Prop.



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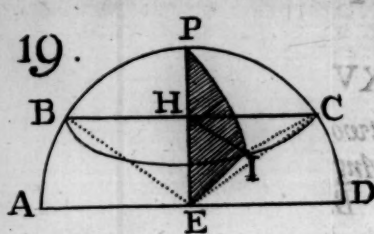
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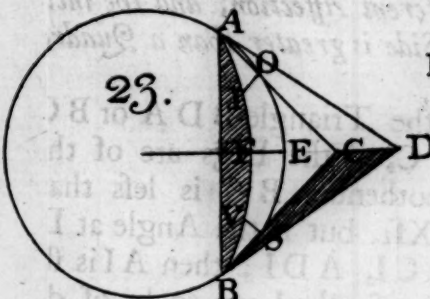


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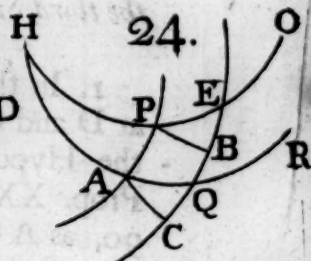
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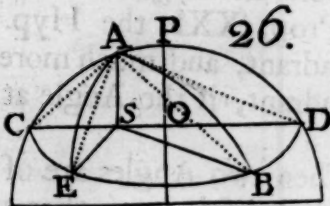
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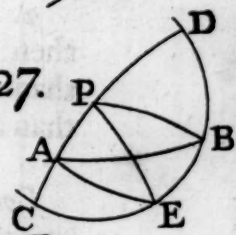
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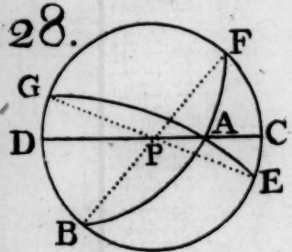
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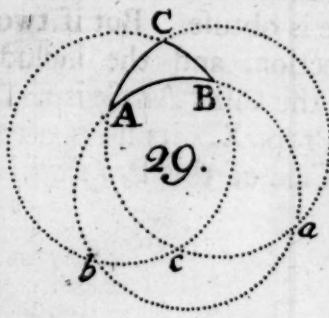
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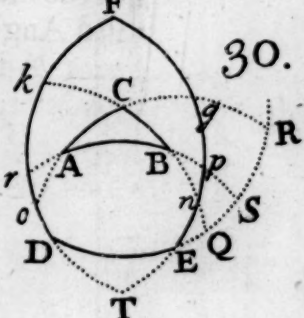
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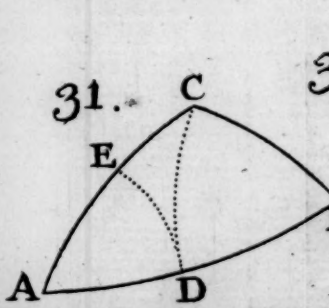
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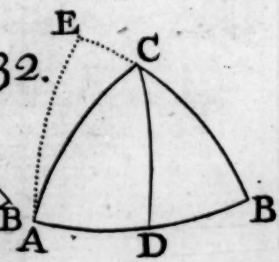
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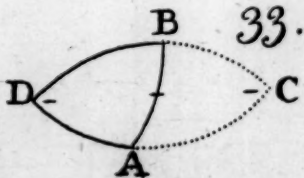
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32.



33.



S E C T. III.

The Proportions for calculating the Sides and Angles of Spherical Triangles.

P R O P. XXVI.

In any right-angled spherical Triangle ABC, if the Angle at A be given, and the Side AC, which is adjacent to it, be given, to find the Side AB, which is opposite to it, and the Angle at C.

Let the Triangle BAC be right-angled at A: draw AD, perpendicular to the Base BC, from the Vertex A to the Base BC, and draw BH, perpendicular to AD, from the Vertex B to AD. Then the Triangle BAC is right-angled at A, and the Triangle BAH is right-angled at A, and the Triangle BHC is right-angled at H. Also the Angle BAC is given, and the Side AC is given, and the Side AB is required.

Since the Angle BAC is given, the Angle ABC is also given, and the Side AC is given, the Side AB may be found by the Rule of Sines.

As the Sine of the Angle BAC is to the Sine of the Angle ABC, so the Side AC is to the Side AB.

Therefore the Side AB may be found by the Rule of Sines.

S E C T. III.

The Proportions for calculating the Sides and Angles of spherical Triangles.

P R O P. XXVI.

FIG.

40.

In any right angled spherical Triangle ABC, it is,

As Radius :

To sine of the Hypotenuse BC ::

So Sine of any Angle C :

To Sine of its opposite Side AB.

Let the Triangle BAC be right angled at A; draw AG, CG to the Center G of the Sphere. From B let fall BI $\perp$ to AG, and thro' I draw IH $\perp$ to GC; and draw BH, which will be in the Plane of the Circle BC. Since the Plane ABG is $\perp$ ACG, therefore BI is $\perp$ ACG, and CH $\perp$ to the Plane BIH. Then BI is the Sine of BA, and BH the Sine of BC, and by Cor. 4. Prop. IV. $\angle BHI =$ spherical Angle ACB.

This done, in the right angled plain Triangle BIH, it will be (by Prop. III. B. II.) Rad : BH :: S.H : BI, that is Rad : S.BC :: S.C : S.AB.

Cor. 1. In right angled Triangles that have equal Angles at the Base; the Sines of the Hypotenuses are as the Sines of the Perpendiculars.

For if $\angle C$ be given, the S.BC is to the S.AB in a given Ratio.

Cor. 2. In right angled Triangles having equal Hypotenuses; the Sines of the Perpendiculars are as the Sines of their opposite Angles. For then BC is given.

Cor.

FIG. *Cor.* 3. In a right angled Triangle A B F, 2 Cof.
 41. Hyp. A F $\times$ Rad = Cof. Sum of the Sides A B, B F
 + Cof. their Difference.

For in the Triangle E D F,
 Rad : S. E F :: S. E : S. D F, that is Rad : Cof. B F ::
 Cof. A B : Cof. A F, whence Rad $\times$ Cof. A F = Cof.
 A B $\times$ Cof. B F = (by Sch. 4. I.) Cof. $\overline{A B + B F} + \text{Cof.}$
 $\overline{A B - B F}$: $\times$ into $\frac{1}{2}$ Radius.

P R O P. XXVII.

40. In any right angled Triangle A B C, as
 Radius:
 Sine of one Side A C ::
 Tan. Angle adjacent C :
 Tan. opposite Side A B.

Supposing the same Construction as in the last
 Prop. then in the right angled plane Triangle H I G
 it will be (by Prop. III. B. II.) Rad : S. A G C or A C
 :: I G : I H. And in the right angled Triangle
 B I G (right angled at I) Rad : Tan. B G I or A B ::
 I G : I B. And in the right angled plane Triangle
 B I H, Tan. B H I or C : Rad :: I B : I H. There-
 fore *ex equo*, Tan C : Tan. A B :: I G : I H; that is
 as Rad : S. A C :: Tan. C : Tan. A B.

Cor. 1. In right angled Triangles having equal
 Angles at the Base; the Sines of the Bases are as the
 Tangents of the Perpendiculars.

For S. A C is to Tan. A B in a given Ratio.

Cor. 2. In right angled Triangles having equal
 Bases; the Tangents of the Perpendiculars are as the
 Tangents of the opposite Angles.

For if A C be given, these are in a given Ratio.

P R O P.

In any right angled or quadrantal Triangle; the Sine of middle Part and Radius, are reciprocally proportional to the Tangents of the Extrems conjunct, and to the Cosines of the Extrems disjunct.

41.

This in the Lord *Neper's* Theorem, and is demonstrated by Induction thus.

Produce the Sides *AB*, *AF*, *BF* to Quadrants, and describe the Arch *CDE*. Then since the Arches *AC*, *AD*, *BE*, *CE* are Quadrants, the Angles *B*, *C*, *D* will be right; and *ED* = Complement of *DC* = Comp. *A*; and *FD* = Comp. *AF*; and *EF* = Comp. *FB*. Also the Cosine of the Complement is the Sine, and the Cotan. of the Comp. is the Tangent.

Now the middle Part must be either a Leg, the Comp. of an Angle, or the Comp. of the Hypothenuse.

1. Suppose a Leg as *AB* be middle Part, then *BF* and Comp. $\angle A$ are Extrems conjunct. But by Prop XXVII. $\text{Rad} : S. AB :: \text{Tan. } A : \text{Tan. } FB$. Or thus $S. AB : \text{Tan. } FB :: (\text{Rad} : \text{Tan. } A ::) \text{Cotan. } A : \text{Radius}$, by Cor. 2. Prop. 1. B. I.

2. Let Comp. of an Angle *A* be middle Part, then *AB* and Comp. *AF* are Extrems conjunct; and in the Triangle *DEF*, by Prop. XXVII. $\text{Rad} : S. ED :: \text{Tan. } E : \text{Tan. } DF$; or $S. ED : \text{Tan. } DF :: (\text{Rad} : \text{Tan. } E ::) \text{Cotan. } E : \text{Radius}$, by Cor. 2. Pr. 1. I. that is $\text{Cof. } A : \text{Cotan. } AF :: \text{Tan. } AB : \text{Rad}$.

3. Let the Comp. Hypothenuse *AF* be middle Part, then Comp. *A* and Comp. *F* are Extrems conjunct. Therefore in the Triangle *DEF*, $\text{Rad} : S. DF :: \text{Tan. } F : \text{Tan. } ED$; or $S. DF : \text{Tan. } ED (:: \text{Rad} : \text{Tan. } F) :: \text{Cotan. } F : \text{Radius}$, that is $\text{Cof. } AF : \text{Cotan. } A :: \text{Cotan. } F : \text{Radius}$.

Therefore $\text{Radius} \times S. \text{middle Part} = \text{Rectangle of the Tangents of the Extrems conjunct}$. Again

1. Let a Leg *AB* be middle Part, then Comp. *F*, and Comp. *AF* are Extrems disjunct, therefore by Prop. XXVI. $\text{Rad} : S. AF :: S. F : S. AB$.

2. Let

FIG. 41. 2. Let Comp. of an Angle A be middle Part; then FB and Comp. F are Extrems disjunct: Therefore in the Triangle DEF, by Prop. XXVI. Rad : S.EF :: S.F : S.ED, that is Rad : Cof. BF :: S.F : Cof. A.

3. Let Comp. Hyp. AF be middle Part; then AB, FB are Extrems disjunct. And in the Triangle DEF, by Prop. XXVI. Rad : S.EF :: S.E : S.DF, that is Rad : Cof. BF :: Cof. AB : Cof. AF.

Therefore Radius $\times$ Sine of middle Part = Rect-angle of the Cofines of the Extrems disjunct.

And in a quadrantal Triangle, its supplemental one, by Prop. X. will be a right angled Triangle. And since this Prop. holds in that right angled Triangle, it will equally hold in the quadrantal Triangle, wherein the Parts are the Supplements of the others; because Arches and their Supplements have the same Sines, Cofines, Tangents and Cotangents.

42. Cor. 1. If a Perpendicular FI be let fall on the Base of a Spherical Triangle the Cofines of the Angles at the Base are as the Sines of the Angles at the Vertex.

For in the right angled Triangles AFI, BFI, it is Cof. A : S.AFI :: (Cof. FI : Rad ::) Cof. FBI : S.BFI.

Cor. 2. The Cofines of the Sides are as the Cofines of the Segments of the Base.

For Cof. AI : Cof. AF :: (Rad : Cof. FI ::) Cof. BI : Cof. BF.

Cor. 3. The Sines of the Segments of the Base are as the Cotangents of the Angles at the Base.

For Cotan. A : S.AI :: (Rad : Tan. FI ::) Cot. IBF : S.IB.

Cor. 4. The Cotangents of the Sides, are as the Cofines of the Angles at the Vertex.

For

For Cotan. $AF : \text{Cof. } AFI :: (\text{Rad} : \text{Tan. } FI ::)$ FIG.
 Cotan. $BF : \text{Cof. } BFI$ 42.

Cor. 5. The Tangents of the Segments of the Base are as the Tangents of the Angles at the Vertex.

For Tan. $AFI : \text{Tan. } AI :: (\text{Rad} : S. FI ::)$ Tan.
 $BFI : \text{Tan. } BI$

SCHOLIUM.

This Prop. will resolve all right angled and quadrantal Triangles; and also all oblique ones, (except where 3 $\angle$'s or 3 sides are given,) by letting fall a Perpendicular dividing it into two right angled Triangles.

PROP. XXIX.

In any spherical Triangle, the Sines of the Sides are proportional to the Sines of their opposite Angles. 43.

On AB, AF let fall the Perpendiculars FH, BI; then by Prop. XXVI.

$$S.AF : S.FH :: \text{Rad} : S.A$$

and $S.FH : S.FB :: S.B : \text{Rad}$. therefore
 ex equo $S.AF : S.FB :: S.B : S.A$

again $S.AB : S.BI :: \text{Rad} : S.A$

and $S.BI : S.BF :: S.F : \text{Rad}$. and
 ex equo $S.AB : S.BF :: S.F : S.A$.

Cor. 1. If a great Circle be drawn from the Vertex of a Triangle to cut the Base; the Rectangles of the Sines, of the Sides and of the vertical Angles; are directly as the Sines of the Segments of the Base. 44.

For $S.CDA : S.CA :: S.CAD : S.CD =$
 $\frac{S.CA \times S.CAD}{S.CDA}$, and $S.CDA$ or $BDA : S.AB ::$

$$S.DAB : S.DB = \frac{S.AB \times S.DAB}{S.CDA} \text{ therefore } S.CD :$$

$$S.DB :: S.CA \times S.CAD : S.BA \times S.DAB.$$

Cor.

FIG. Cor. 2. If the vertical Angle of a Triangle be bisected; the Sines of the Segments of the Base are as the Sines of the Sides; $S.CD : S.BD :: S.AC : S.AB$.

P R O P. XXX.

42. In any spherical Triangle, if a Perpendicular be let fall on the Base, it is

As the Sine of the Sum of the Sides, $AF, BF :$

Sine of their Difference ::

Cot. of $\frac{1}{2}$ Sum of the Angles at the Vertex, $AFI, BFI :$

Tan. of half their Difference.

For by Cor. 4. Pr. XXVIII. $\text{Cotan. } AF : \text{Cotan. } BF :: \text{Cof. } AFI : \text{Cof. } BFI$. or by Cor. 4. Pr. I. B. I. $\text{Tan. } AF : \text{Tan. } BF :: \text{Cof. } BFI : \text{Cof. } AFI$. And by Composition and Division, $\text{Tan. } AF + \text{Tan. } BF : \text{Tan. } AF - \text{Tan. } BF :: \text{Cof. } BFI + \text{Cof. } AFI : \text{Cof. } BFI - \text{Cof. } AFI$, that is by Cor. 4 and 2 Pr. VII. B. I. $S.AF + BF : S.AF - BF :: \text{Cotan. } AFI + BFI : \text{Tan. } \frac{AFI + BFI}{2}$.

Cor. 1. If the Perpendicular falls within $\frac{AFI + BFI}{2}$ = half the Angle at the Vertex F : if it falls without $\frac{AFI - BFI}{2}$ = half the vertical Angle.

Cor. 2. As Sine of Sum of the Sides :

Sine of Difference of the Sides ::

Cotan. half the vertical Angle :

Tan. half Difference, or half Sum, of the vertical Angles, according as the Perpendicular falls within or without.

P R O P. XXXI.

42. Let a Perpendicular fall on the Base of a Triangle then
Tan. half the Sum of the Sides, $AF, BF :$
Tan. half their Difference ::

Tan.

Tan. half the Sum of the Angles at the Base, B, A : FIG.
Tan. half their Difference. 42.

For by Prop. XXIX. $S.A.F : S.B.F :: S.B : S.A$.
 And by Composition and Division, $S.A.F + S.B.F : S.A.F - S.B.F :: S.B + S.A : S.B - S.A$; that is by
 Cor. 1. Pr. VII. B. I. $\text{Tan. } \frac{AF+BF}{2} : \text{Tan. } \frac{AF-BF}{2}$
 $:: \text{Tan. } \frac{B+A}{2} : \text{Tan. } \frac{B-A}{2}$.

P R O P. XXXII.

Let a Perpendicular fall on the Base of a spherical 42.
Triangle, then

Cotan. half the Sum of the Angles at the Base, A, B :

Tan. half their Difference ::

Tan. $\frac{1}{2}$ Sum of the Angles at the Vertex, AFI, BFI :

Tan. half their Difference.

For by Cor. 1. Prop. XXVIII. $\text{Cof. } A : \text{Cof. } B :: S.AFI : S.BFI$. And $\text{Cof. } A + \text{Cof. } B : \text{Cof. } A - \text{Cof. } B :: S.AFI + S.BFI : S.AFI - S.BFI$, that is by Cor. 1
 and 2. Prop. 7. I. $\text{Cotan. } \frac{B+A}{2} : \text{Tan. } \frac{B-A}{2} :: \text{Tan. } \frac{AFI+BFI}{2} : \text{Tan. } \frac{AFI-BFI}{2}$.

Note, If the Perpendicular falls within, $\frac{AFI+BFI}{2}$
 $=$ half the vertical Angle F; but $\frac{AFI-BFI}{2}$, if it
 fall without.

Cor. Cotan. half the Sum of the Angles at the Base :

Tan. half their Difference ::

Tan. half the vertical Angle :

*Tan. half the Difference, or half the Sum of the
 vertical Angles, as the Perpendicular falls
 within or without.*

L

This

FIG.

42.

This is plain when the Perpendicular falls within ;
and when it falls without, $+BFI$ becomes $-BFI$,
and the contrary.

P R O P. XXXIII.

42.

Suppose a Perpendicular to fall on the Base of a spherical Triangle, then as

Tan. half Sum of the Segments of the Base AI, BI :

Tan. half the Sum of the Sides, AF, BF ::

Tan. half Difference of the Sides AF, BF :

Tan. $\frac{1}{2}$ Diff. of the Segments of the Base, AI, BI.

For by Cor. 2. Prop. XXVIII. $\text{Cof. BF} : \text{Cof. AF} :: \text{Cof. BI} : \text{Cof. AI}$. And $\text{Cof. BF} + \text{Cof. AF} : \text{Cof. BF} - \text{Cof. AF} :: \text{Cof. BI} + \text{Cof. AI} : \text{Cof. BI} - \text{Cof. AI}$;
that is by Cor. 2. Prop. VII. B. I. $\text{Cotan. } \frac{\text{AF} + \text{BF}}{2} : \text{Tan. } \frac{\text{AF} - \text{BF}}{2} :: \text{Cotan. } \frac{\text{AI} + \text{BI}}{2} : \text{Tan. } \frac{\text{AI} - \text{BI}}{2}$.

Or by Cor. 4. Prop. I. B. I. $\text{Tan. } \frac{\text{AI} + \text{BI}}{2} : \text{Tan. } \frac{\text{AF} + \text{BF}}{2} :: (\text{Cotan. } \frac{\text{AF} + \text{BF}}{2} : \text{Cotan. } \frac{\text{AI} + \text{BI}}{2} ::) \text{Tan. } \frac{\text{AF} - \text{BF}}{2} : \text{Tan. } \frac{\text{AI} - \text{BI}}{2}$.

Here as the Perpendicular falls within or without, $\text{AI} + \text{BI}$ or $\text{AI} - \text{BI}$ will be the Base.

Cor. 1. As Tan. half the Base :

Tan. half Sum of the Sides ::

Tan. half Difference of the Sides :

Tan. half the alternate Base.

Where the alternate Base is half the Difference or half the Sum of the Segments of the Base, according as the Perpendicular falls within or without.

Cor. 2. In a right angled Triangle ; the Rectangle of the Tan. of half the Hypothenuse + half one Leg, and

and the Tan. half the Hypothenuſe — half that Leg FIG.
= Tan. Square of half the other Leg, For then $BI=0$.

P R O P. XXXIV.

Let fall a Perpendicular upon the Baſe of a Triangle, 42.
then

Sine of the Sum of the Angles at the Baſe B, A :

Sine of their Difference ::

Tan. half Sum of the Segments of the Baſe, AI, BI :

Tan. half their Difference.

For by Cor. 3. Prop. XXVIII. $\text{Cotan. } A : \text{Cotan. } B :: S.AI : S.BI$, or $\text{Tan. } B : \text{Tan. } A :: S.AI : S.BI$.
And $\text{Tan. } B + \text{Tan. } A : \text{Tan. } B - \text{Tan. } A :: S.AI + S.BI : S.AI - S.BI$; that is by Cor. 4 and 1, Pr. VII.

$$B. I. \frac{S.B+A}{2} : \frac{S.B-A}{2} :: \text{Tan. } \frac{AI+BI}{2} : \text{Tan. } \frac{AI-BI}{2}.$$

According as the Perpendicular falls within or without, $\frac{AI+BI}{2}$, or $\frac{AI-BI}{2}$ will be equal to half the Baſe. Whence

Cor. 1. *Sine of the Sum of the Angles at the Baſe :*

Sine of their Difference ::

Tan. half the Baſe :

Tan. half the alternate Baſe. (ſee Cor. 1. Pr. 33.)

For when the Perpendicular falls without, $+BI$ becomes $-BI$, and the contrary.

P R O P. XXXV.

Let fall a Perpendicular on the Baſe, then as

Sine of the Sum of the Segments of the Baſe, AI, BI :

Sine of their Difference ::

Sine of the Sum of the Angles at the Vertex, AFI, BFI :

Sine of their Difference.

L 2

For

FIG. 42. For by Cor. 5. Prop. XXVIII. $\text{Tan. AI} : \text{Tan. BI} :: \text{Tan. AFI} : \text{Tan. BFI}$. And $\text{Tan. AI} + \text{Tan. BI} : \text{Tan. AI} - \text{Tan. BI} :: \text{Tan. AFI} + \text{Tan. BFI} : \text{Tan. AFI} - \text{Tan. BFI}$. that is by Cor. 4. Pr. VII. $\text{B.I. S.AI} + \text{BI} : \text{S.AI} - \text{BI} :: \text{S.AFI} + \text{BFI} : \text{S.AFI} - \text{BFI}$. Here, as the Perpendicular falls within or without; $\text{AI} + \text{BI}$, or $\text{AI} - \text{BI}$ is the Base; and $\text{AFI} + \text{BFI}$, or $\text{AFI} - \text{BFI}$, the vertical Angle. Whence

Cor. As Sine of the Base :

Sine of the vertical Angle ::

So Sine of the diff. Segments of the Base :

Sine of the diff. Angles at the Vertex, when the Perpendicular falls within ::

Or Sine of the Sum of the Segments of the Base :

Sine of the Sum of the vertical Angles, when the Perpendicular falls without.

PROP. XXXVI.

42. As Cosine of half the Sum of two Sides, AF, BF :
Cosine of half their Difference ::
Cotan. half the included Angle AFB :
Tan. half Sum of the opposite Angles A, ABF .

Put $\text{AF} + \text{BF} = \text{P}$, $\text{AF} - \text{BF} = p$, $\text{ABF} + \text{A} = \text{Q}$,
 $\text{ABF} - \text{A} = q$. $\text{AI} + \text{BI} = \text{R}$, $\text{AI} - \text{BI} = r$, $\text{AFI} + \text{BFI} = \text{V}$, $\text{AFI} - \text{BFI} = v$.

Then since the Tangents are reciprocally as the Cotangents, we shall have by

Prop. 30. $\text{SP} : \text{Sp} :: \text{Cotan. } \frac{1}{2}\text{V} : \text{T. } \frac{1}{2}v$.

31. $\text{Cot. } \frac{1}{2}\text{P} : \text{Cot. } \frac{1}{2}p :: \text{T. } \frac{1}{2}q : \text{T. } \frac{1}{2}\text{Q}$

32. $\text{T. } \frac{1}{2}q : \text{T. } \frac{1}{2}v :: \text{Cot. } \frac{1}{2}\text{V} : \text{T. } \frac{1}{2}\text{Q}$.

And compounding these Ratios, and expunging equal Quantities.

$\text{S.P} \times \text{Cot. } \frac{1}{2}\text{P} : \text{S.p} \times \text{Cot. } \frac{1}{2}p :: \text{Cot. Square, } \frac{1}{2}\text{V} : \text{Tan.}$

Tan. Square $\frac{1}{2}Q$. But by Sch. Pr. 2. I. $\frac{2\text{Cof.}^2\frac{1}{2}P}{\text{Cot.}\frac{1}{2}P}$ FIG. 42.

$$= S.P, \text{ or } \text{Cof.}^2\frac{1}{2}P = \frac{S.P \times \text{Cot.}\frac{1}{2}P}{2}, \text{ and } \text{Cof.}^2\frac{1}{2}p =$$

$$\frac{S.p \times \text{Cot.}\frac{1}{2}p}{2}, \text{ therefore Cof. Square } \frac{1}{2}P : \text{Cof. Square}$$

$$\frac{1}{2}p :: \text{Cot. Square } \frac{1}{2}V : \text{T. Square } \frac{1}{2}Q; \text{ and Cof. } \frac{1}{2}P$$

$$: \text{Cof. } \frac{1}{2}p :: \text{Cot. } \frac{1}{2}V : \text{Tan. } \frac{1}{2}Q.$$

This is plain when the Perpendicular falls within; and when it falls without; instead of the third Proportion, substitute this (by Cor. Pr. XXXII.) $T.\frac{1}{2}q : T.\frac{1}{2}V :: \text{Cot. } \frac{1}{2}v : T.\frac{1}{2}Q$ and you will have at last, this; $\text{Cof. } \frac{1}{2}P : \text{Cof. } \frac{1}{2}p :: \text{Cot. } \frac{1}{2}v : \text{Tan. } \frac{1}{2}Q$.

Cor. 1. As *Sine* of half the Sum of the Sides :

Sine of half their Difference ::

Cotan. half the included Angle :

Tan. half Difference of the opposite Angles.

For by the former Proportions, a little vary'd,

$$S.P : S.p :: \text{Cot. } \frac{1}{2}V : T.\frac{1}{2}v$$

$$T.\frac{1}{2}P : T.\frac{1}{2}p :: T.\frac{1}{2}Q : T.\frac{1}{2}q$$

$$T.\frac{1}{2}Q : T.\frac{1}{2}v :: \text{Cot. } \frac{1}{2}V : T.\frac{1}{2}q$$

then proceeding as before, it will be found (by Sch. 2. Prop. 2. B. I.) that

$S.\frac{1}{2}P : S.\frac{1}{2}p :: \text{Cot. } \frac{1}{2}V : T.\frac{1}{2}q$. when the Perpendicular falls within; and when it falls without, instead of the third Proportion put this (by Cor. Prop. XXXII.) $\text{Cot. } \frac{1}{2}V : \text{Cot. } \frac{1}{2}v :: \text{Cot. } \frac{1}{2}Q : T.\frac{1}{2}q$ and you will get at last $S.\frac{1}{2}P : S.\frac{1}{2}p :: \text{Cot. } \frac{1}{2}v : \text{Tan. } \frac{1}{2}q$.

Cor. 2. *Cof.* half Sum of two Sides :

Cof. half their Difference ::

Tan. half Difference of their opposite Angles :

Tan. half Difference, or half Sum of the vertical Angles; according as the Perpendicular, on the third Side, falls within or without.

This follows from the present Prop. and Cor. Prop. XXXII.

FIG.

42.

PROP. XXXVII.

*As Cos. half Sum of two Angles A, ABF:**Cos. half their Difference ::**Tan. half the included Side AB:**Tan. half Sum of the opposite Sides, AF, BF.*

Supposing as in the last Prop. then by

$$\text{Prop. 31. } \text{Cot. } \frac{1}{2} Q : \text{Cot. } \frac{1}{2} q :: \text{T. } \frac{1}{2} p : \text{T. } \frac{1}{2} P$$

$$33. \text{T. } \frac{1}{2} p : \text{T. } \frac{1}{2} r :: \text{T. } \frac{1}{2} R : \text{T. } \frac{1}{2} P$$

$$34. \text{S.Q} : \text{S.q} :: \text{T. } \frac{1}{2} R : \text{T. } \frac{1}{2} r$$

But (by Sch. Pr. II. B. I.) $\frac{1}{2} \text{S.Q} \times \text{Cot. } \frac{1}{2} Q = \overline{\text{Cof.}}^2 \frac{1}{2} Q$,
and $\frac{1}{2} \text{S.q} \times \text{Cot. } \frac{1}{2} q = \overline{\text{Cof.}}^2 \frac{1}{2} q$, therefore by com-
pounding,

$$\overline{\text{Cof.}}^2 \frac{1}{2} Q : \overline{\text{Cof.}}^2 \frac{1}{2} q :: \text{T.}^2 \frac{1}{2} R : \text{T.}^2 \frac{1}{2} P.$$

$$\text{Or } \text{Cof. } \frac{1}{2} Q : \text{Cof. } \frac{1}{2} q :: \text{T. } \frac{1}{2} R : \text{T. } \frac{1}{2} P.$$

But when the Perpendicular falls without, instead
of the third Proportion, put this (by Cor. Prop.
XXXIV), $\text{S.Q} : \text{S.q} :: \text{T. } \frac{1}{2} r : \text{T. } \frac{1}{2} R$. Whence at
last you will have this

$$\text{Cof. } \frac{1}{2} Q : \text{Cof. } \frac{1}{2} q :: \text{T. } \frac{1}{2} r : \text{T. } \frac{1}{2} P.$$

*Cor. 1. Sine half Sum of two Angles, A, ABF:**Sine of half their Difference ::**Tan. half the included Side, AB:**Tan. half Diff. of the opposite Sides, AF, BF.*

For by varying the Terms of the foregoing Pro-
portions, it will be

$$\text{T. } \frac{1}{2} Q : \text{T. } \frac{1}{2} q :: \text{T. } \frac{1}{2} P : \text{T. } \frac{1}{2} p.$$

$$\text{T. } \frac{1}{2} P : \text{T. } \frac{1}{2} r :: \text{T. } \frac{1}{2} R : \text{T. } \frac{1}{2} p.$$

$$\text{S.Q} : \text{S.q} :: \text{T. } \frac{1}{2} R : \text{T. } \frac{1}{2} r.$$

but (by Sch. Prop. 2. I.) $\frac{1}{2} \text{S.Q} \times \text{T. } \frac{1}{2} Q = \text{S}^2 \frac{1}{2} Q$, and
 $\frac{1}{2} \text{S.q} \times \text{T. } \frac{1}{2} q = \text{S}^2 \frac{1}{2} q$, therefore by compounding,

$$\text{S. } \frac{1}{2} Q : \text{S. } \frac{1}{2} q :: \text{T. } \frac{1}{2} R : \text{T. } \frac{1}{2} p.$$

If

If the Perpendicular fall without, instead of the FIG. 42.
 third Proportion put this (by Cor. Prop. XXXIV.)
 $S.Q : S.q :: T.\frac{1}{2}r : T.\frac{1}{2}R$ which will give this
 Proportion, $S.\frac{1}{2}Q : S.\frac{1}{2}q :: T.\frac{1}{2}r : T.\frac{1}{2}p$.

Cor. 2. *Cof.* $\frac{1}{2}$ Sum of the Angles at the Base, A, ABF:

Cof. half their Difference ::

Tan. half Diff. opposite Sides, AF, BF:

Tan. half Diff. or half Sum, of the Segments of the
 Base; according as the Perpendicular falls within
 or without.

This follows from the present Prop. and Cor. 1.
 Prop. XXXIII.

P R O P. XXXVIII.

In any spherical Triangle AFB; the Rectangle of the
 Sines of any two Sides $\times$ *Cof.* included Angle + the Rect-
 angle of their *Cosines* $\times$ Radius = *Cof.* of the third Side
 $\times$ Radius Square: that is $S.AB \times S.FB \times \text{Cof. } B +$
 $\text{Cof. } AB \times \text{Cof. } FB \times \text{Rad} = \text{Cof. } AF \times \text{Rad}^2$.

Let $a = S.AB$, $m = \text{Cof. } AB$, $s = S.B$.

$b = S.FB$, $n = \text{Cof. } FB$, $c = \text{Cof. } B$, $r = \text{Cot. } B$.

$r = \text{Rad.}$

$p = \sqrt{rrnn + bbcc}$.

By Prop. 1. I. $n : b :: r : \text{Tan. } FB :: \text{Cot. } FB$
 $: r ::$ (by Prop. XXVIII.) $c : \text{Tan. } BI = \frac{cb}{n}$. And

by Sch. Pr. 1. I. Sine of $BI = \frac{rcb}{p}$, and *Cof.* $BI =$

$\frac{rrn}{p}$; and by Cor. Pr. 6. I. *Cof.* $AI = \frac{abc + mnr}{p}$,

but by Cor. 2. Prop. XXVIII. $\frac{rrn}{p} : \frac{abc + mnr}{p} ::$

$n : \text{Cof. } AF = \frac{abc + mnr}{rr}$

Cor. 1. The *Cofine* of any Angle of a spherical Tri-
 angle is equal to the *Cofine* of the opposite Side $\times$
 Radius Square — the Rectangle of the *Cosines* of the
 L 4 includ-

FIG. 42. including Sides $\times$ Radius, and the whole divided by the Rectangle of the Sines of the including Sides.

$$\text{For Cof. B} = \frac{\text{Cof. AF} \times rr - \text{Cof. AB} \times \text{Cof. FB} \times r}{\text{S. AB} \times \text{S. FB}}$$

$$\text{Cor. 2. Cotan. A} = \frac{\text{S. AB} \times \text{Cof. FB} \times \text{Rad.} - \text{S. FB} \times \text{Cof. AB} \times \text{Cof. B}}{\text{S. B} \times \text{S. FB}}$$

$$\text{For by Prop. 6. I. S. AI} = \frac{anr - bcm}{p}, \text{ and by Cor.}$$

$$3. \text{ Prop. XXVIII. } \frac{rcb}{p} : \frac{anr - bcm}{p} :: r : \text{Cot. A} = \frac{anr - bcm}{rcb} r = \frac{anr - mbc}{bs}, \text{ because } \frac{r}{c} = \frac{r}{s} \text{ by Sch. Pr. 1. I.}$$

$$\text{Cor. 3. } \frac{\text{S. AB} - \text{FB} \times rr + \text{S. FB} \times \text{Cof. AB} \times \text{vers. B}}{\text{S. B} \times \text{S. FB}} = \text{Cotan. A.}$$

For since $\text{Cof. B} = r - \text{vers. B}$, therefore by Cor. 2.

$$\text{Cot A} = \frac{anr - mbr + mb \times \text{vers. B}}{bs}; \text{ but by Prop. 6. I.}$$

$$an - bm = \text{S. AB} - \text{FB} \times r, \text{ whence the Cor. is evident.}$$

SCHOLIUM.

If the Angle ABF be greater than a right Angle, then its Cosine c will be negative.

PROP. XXXIX.

45. The Rectangle of the Sines of two Angles $\times$ Cosine of the included Side divided by Radius Square — the Rectangle of their Cosines divided by Radius, is equal to the Cosine of the third Angle.

$$\text{that is, } \frac{\text{S. A} \times \text{S. B} \times \text{Cof. AB}}{rr} - \frac{\text{Cof. A} \times \text{Cof. B}}{r} = \text{Cof. F.}$$

Let

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Let fall the Perpendicular BR, and put **FIG.**

45.

$S.A=a$, $Cof.A=m$, $Cot.A=r$, $Cof.AB=c$, $Rad=r$.

$S.B=b$, $Cof.B=n$, $q=\sqrt{rr+cc}$

then by Prop. XXVIII. $Cot. ABR=\frac{rc}{r}$, and (by

Sch. 1. I.) its Sine $=\frac{r\tau}{q}$, and its Cosine $=\frac{rc}{q}$. And

by Prop. 6. I. $S.RBF=\frac{bc-n\tau}{q}$; and by Cor. 1. Pr.

XXVIII. $\frac{r\tau}{q} : \frac{bc-n\tau}{q} :: m : Cof. F = \frac{bc-n\tau}{r\tau} m =$

$\frac{bcm}{r\tau} - \frac{mn}{r} = \frac{abc}{rr} - \frac{mn}{r}$; because $\frac{m}{\tau} = \frac{a}{r}$, by Sch. 1. I.

Cor. 1. The *Cosine* of any Side is equal to the Rect-angle of the *Cosines* of the including Angles $\times$ *Radius* + *Cof.* opposite Angle $\times$ *Radius* square, and the whole divided by the Rectangle of the *Sines* of the including Angles.

For c or the *Cof.* $AB = \frac{mnr+rr \times Cof. F}{ab}$.

Cor. 2. $Cot. FB = \frac{S.B \times Cot. A}{S.AB} + \frac{Cof. B \times Cot. AB}{r}$.

For let $s=S.AB$, $t=Cotan. AB$, then by Cor. 1.

Prop. 6. I. $Cof. RBF = \frac{b\tau+nc}{q}$, and by Cor. 4. Prop.

XXVIII. $\frac{rc}{q} : \frac{b\tau+nc}{q} :: t : Cot. BF = \frac{b\tau+nc}{rc} t = \frac{b\tau}{s} +$

$\frac{n\tau}{r}$, because $ts=cr$. Or $Cot. BF = \frac{rbm+nac}{sa}$, be-

cause $\tau = \frac{mr}{a}$, and $t = \frac{cr}{s}$, by Sch. Prop. 1. I.

Cor. 3. $Cotan. FB =$

$\frac{r^2 \times S.A + B - S.A \times Cof.B \times verf.AB}{S.A \times S.AB}$.

For

FIG. For let $v = \text{vers. } AB = r - c$, and $c = r - v$, then Cot.

45. $FB = \frac{rbm + nac}{sa} = \frac{rbm + ran - anv}{sa}$; but (by Pr. 5. I.) $mb + an = r \times S. A + B$, whence $\text{Cot. } BF = \frac{rr \times S. A + B - anv}{as}$.

PROP. XL.

47. Let a, s, b, p, f, d be the Sines, } of the Angles and
and m, c, n, i, g, e the Cosines }
Sides, as mark'd in the Figure. And
 $bb = bb - aa = mm - nn$.
and $kk = 1 - bbpp = 1 - aadd$.

Radius = 1. Then

$$\text{Sine of } \angle F = \frac{bb}{bim - aen} = \frac{bim + aen}{kk}.$$

For 1. $bif - ng = m$. By Prop. XXXIX.

2. $aef - mg = n$. By Prop. XXXIX.

3. $\frac{n}{b} = \frac{pe - dig}{df}$. By Cor. 2. Pr. XXXVIII.

For by Pr. 1. I. $\frac{b}{n} = \text{Cotan. } B$. From the two first

Equations $g = \frac{bif - m}{n} = \frac{aef - n}{m}$, whence $bifm - m^2 = aefn - nn$, and $f = \frac{m^2 - n^2}{bim - aen}$.

Again, by the third Equation $dnf = bpe - bdig = ade - bdig$, because (by Prop. XXI.) $bp = ad$; and $nf = ae - big$; whence $g = \frac{ae - nf}{bi} = \frac{bif - m}{n}$, and $aen - fn^2 = b^2 i^2 f - bim$, whence $f = \frac{aen + bim}{n^2 + bbii} = \frac{aen + bim}{1 - bbpp}$; because $ii = 1 - pp$.

Cor. 1. $\text{Cof. } F = \frac{abei - mn}{kk} = \frac{aem - bin}{bim - aen}$.

For

For by the 1st and 3^d Equations $f = \frac{ng+m}{bi} = \frac{ae-big}{n}$, FIG. 47.

and $nng+mn=aebi-bbiig$; whence $g = \frac{aebi-mn}{nn+bbii}$
 $= \frac{aebi-mn}{1-bbpp}$.

Again, by the 1st and 2^d Equations $f = \frac{m+ng}{bi} =$
 $\frac{n+mg}{ae}$, and $aem+aeng=bin+bing$, whence $g =$
 $\frac{aem-bin}{bim-aen}$.

Cor. 2. Cotan. $F = \frac{ame-bin}{bb} = \frac{e \times S.2A - i \times S.2B}{\text{Cof. } 2A - \text{Cof. } 2B}$

For multiplying the 1st Equation by n , and the 2^d by m , we have $mn=binf-nng=aefm-mmng$, and

$$m^2-n^2.g=aem-bin.f, \text{ and } \frac{g}{f} = \frac{aem-bin}{m^2-n^2} =$$

$$\frac{2aem-2bin}{2b^2-2a^2}; \text{ but (by Prop. 2. I.) } 2am=S.2A, \text{ and } 2bn=S.2B,$$

also $2aa=\text{vers. } 2A$, and $2b^2=\text{vers. } 2B$,
 and $2bb-2aa=\text{vers. } 2B-\text{vers. } 2A=\text{Cof. } 2A-\text{Cof. } 2B$,
 and $b^2-a^2=m^2-n^2$, for $bb+n^2=1=m^2+a^2$.

SCHOLIUM.

Any of the Quantities a, b, p, d , or m, n, i, e , may be expung'd by help of the Equation $bp=ad$, or the Tangents or Cotangents substituted for them.

PROP. XLI.

Suppose as in the half Prop. and let

$$ll=dd-pp=ii-ee,$$

$$tt=1-bbpp=1-aadd$$

Radius = 1. Then

$$\text{Sine of } AB = \frac{ll}{mid-pen} = \frac{mid+pen}{tt}.$$

For

FIG. For 1. $pns + ci = e$, by Prop. XXXVIII.

47. 2. $mds + ce = i$, by Prop. XXXVIII.

3. $\frac{i}{p} = \frac{bm + anc}{as}$, by Cor. 2. Prop. XXXIX.

By the two first Equations $c = \frac{e - pns}{i} = \frac{i - mds}{e}$,

and $ee - pens = ii - imds$, and $imds - pens = ii - ee$,
whence $s = \frac{ii - ee}{dim - pen}$.

Also by the third Equation $asi = pbm + pnac =$
(since $ad = bp$) $adm + pnac$, that is $si = dm + pnc$,
whence $c = \frac{si - dm}{pn} = \frac{e - pns}{i}$, and $si^2 - dim = pen -$

p^2n^2s , whence $s = \frac{pen + dim}{i^2 + p^2n^2} = \frac{pen + dim}{1 - ppbb}$.

Cor. 1. Cof. A B = $\frac{ei - mndp}{tt} = \frac{pin - dem}{pen - dim}$.

For by 1st and 3^d Equations, $s = \frac{e - ci}{pn} = \frac{dm + cpn}{i}$, and

$ei - ci^2 = pndm + p^2n^2c$, whence $c = \frac{ei - pdmn}{i^2 + p^2n^2} = \frac{ei - pdmn}{1 - bbpp}$.

Also by the 1st and 2^d Equations $s = \frac{e - ci}{pn} = \frac{i - ce}{md}$,

and $mde - mdci = pni - pnec$, whence $c = \frac{dem - pin}{dim - pen}$.

Cor. 2. Cotan. A B = $\frac{med - pin}{ll} =$

$\frac{m \times S.2 AF - n \times S.2 BF}{\text{Col. 2 BF} - \text{Col. 2 AF}}$.

For multiplying the 1st Equation by i , and the 2^d
by e , we have $pnsi + cii = ei = mdse + cee$, and $cii -$
 $cee = mdes - pins$, whence $\frac{c}{s} = \frac{med - pin}{ii - ee} = \frac{2med - 2pin}{2dd - 2pp}$;
but by Prop. 2. I. $2ed = S.2 AF$, and $2pi = S.2 BF$, and
 $2dd = \text{vers. } 2 AF$, and $2p^2 = \text{vers. } 2 BF$, and $2dd -$
 $2pp = \text{vers. } 2 AF - \text{vers. } 2 BF = \text{Cof. } 2 BF - \text{Col. } 2 AF$.

SCHO.

SCHOLIUM.

Any of the Quantities a, d, b, p ; or m, e, i, n , may be exterminated by help of the Equation $ad=bp$, or the Tangents or Cotangents substituted instead of them.

PROP. XLII.

In any spherical Triangle, ABF,
As Rectangle of the Sines of the Sides AB, FB :

46.

Radius Square ::

Vers. Base — vers. difference of the Sides :

Vers. included Angle B ::

Cof. difference of the Sides — Cof. Base :

Vers. included Angle B.

Let $a=S.AB$, $m=Cof. AB$, $c=Cof. B$. $v=vers. B$.

$b=S.FB$, $n=Cof. FB$, $d=Cof. AF$, Rad $=r$.

Then (by Cor. 2. Prop. XXXVIII.) $\frac{d^2 - mnr}{ab} =$

$c=r-v$, and $v = \frac{abr - drr + mnr}{ab}$; but (by Cor. 1.

Prop. 6. I.) $\frac{ab+mn}{r} = r \times Cof. AB-FB$. Therefore

$v = \frac{rr \times Cof. AB-FB - drr}{ab} = \frac{vers. AF - vers. AB-FB}{ab}$

$\times rr$.

Cor. 1. Rectangle of the Sines of the Sides :

S. half Sum $\times$ S. half diff. of the Base and diff. Sides ::

And vers. Sum of the Sides — vers. their difference :

vers. Base — vers. difference of the Sides ::

And Cof. difference — Cof. Sum of the Sides :

Cof. diff. Sides — Cof. Base ::

So twice Radius :

vers. Angle included, B.

For let the greater Side $AB=a$, lesser $BF=e$, Base $AF=b$, $a+e=z$, $a-e=d$, $v=vers. B$.

Then

FIG.

Then by this Prop. $S.a \times S.e : \text{vers. } b - \text{vers. } d \times r ::$ 46. $r : v$. But (by Cor. 3. Pr. 3. I.) $\frac{1}{2}r \times \text{vers. } b - \text{vers. } d$ $= S. \frac{b+d}{2} \times S. \frac{b-d}{2}$, and (by Cor. 2. Pr. 3. I.) $S.a \times S.e$ $= \frac{1}{2}r \times \text{Cof. } d - \text{Cof. } z$. Whence you have these two Proportions,

$$S.a \times S.e : S. \frac{b+d}{2} \times S. \frac{b-d}{2} :: 2r : v.$$

And $\text{vers. } z - \text{vers. } d : \text{vers. } b - \text{vers. } d :: 2r : v$.And the Diff. of the versed Sines is $=$ diff. Cofines.

Cor. 2. Rectangle of the Sines of the Sides :

Radius Square ::

Vers. Sum of the Sides — Vers. Base.

Vers. Sup. of the included Angle B.

For (by Cor. 1.)

$$\text{Vers. } z - \text{vers. } d : \text{vers. } b - \text{vers. } d :: 2r : v.$$

And by Division

$$\text{Vers. } z - \text{vers. } d : \text{vers. } z - \text{vers. } b :: 2r : 2r - v,$$

Or $\text{Vers. } z - \text{vers. } d : 2r :: \text{vers. } z - \text{vers. } b : 2r - v$.

$$\text{And } \frac{\text{vers. } z - \text{vers. } d}{2} \times r : rr :: \text{vers. } z - \text{vers. } b : 2r - v.$$

that is (by Cor. 2. Prop. 3. I.) $S.a \times S.e : rr :: \text{vers. } z - \text{vers. } b : 2r - v$, the vers. Sup. B. and the diff. versed Sines $=$ diff. Cofines. Whence

Cor. 3. Vers. Sum of the Sides — vers. diff. Sides :

Vers. Sum Sides — vers. Base ::

Twice Radius :

Vers. Supplement of the included Angle B.

Cor. 4. As Rectangle of the Sines of the Sides :

Radius Square ::

Or as Sum of the Sines $\times$ diff. Sines, of $\frac{1}{2}$ Sum and $\frac{1}{2}$ diff.

Radius Square :: [Sides :

So $S. \frac{1}{2} \text{ Base} + \frac{1}{2} \text{ diff. Sides} \times S. \frac{1}{2} \text{ Base} - \frac{1}{2} \text{ diff. Sides} :$

Sine Square of half the included Angle ::

So $S. \frac{1}{2} \text{ Sum of the 3 Sides} \times S. \frac{1}{2} \text{ Sum 3 Sides} - \text{Base} :$

Cosine Square of half the included Angle B.

Let

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FIG.

Let $V = \text{Sup. } B$, $s = S. \frac{1}{2}B$, $c = \text{Cof. } \frac{1}{2}B$. The rest as before, then (by Pr. 2. I.) $v = \frac{2ss}{r}$, and $V = \frac{2cc}{r}$, 46.

and (by Prop. 4. I.) $\frac{\text{verf. } b - \text{verf. } d}{2} \times r = S. \frac{b+d}{2} \times S.$

$\frac{b-d}{2}$, and $\frac{\text{verf. } z - \text{verf. } b}{2} \times r = S. \frac{z+b}{2} \times S. \frac{z-b}{2}$; and

(by Cor. 2. Prop. 4. I.) $S.a \times S.e = S. \frac{a+e}{2} \times S. \frac{a-e}{2} \times$

$S. \frac{a+e}{2} - S. \frac{a-e}{2} = S. \frac{1}{2}z + S. \frac{1}{2}d \times S. \frac{1}{2}z - S. \frac{1}{2}d$. Now

by this Prop. $S.a \times S.e : rr :: \text{verf. } b - \text{verf. } d : v$ or $\frac{2ss}{r} :: \frac{\text{verf. } b - \text{verf. } d}{2} \times r$ or $S. \frac{b+d}{2} \times S. \frac{b-d}{2} : ss$.

Again $a+e+b=z+b$, and $\frac{a+e+b}{2} - b = \frac{z-b}{2}$,

and by Cor. 2. $S.a \times S.e : rr :: \text{verf. } z - \text{verf. } b : V$ or $\frac{2cc}{r} :: \frac{\text{verf. } z - \text{verf. } b}{2} \times r$ or $S. \frac{z+b}{2} \times S. \frac{z-b}{2} : cc$.

Hence also, since $d=a-e$,

Cor. 5. As Rectangle of the Sines of the Sides :

Radius Square ::

So $S. \frac{1}{2}$ Sum of 3 Sides—one Side, $\times S. \frac{1}{2}$ Sum—other Sine Square of half the included Angle. [Side :

Cor. 6. From half the Sum of the 3 Sides, subtract the Base, and two Sides separately ; then

As $S.$ half Sum of the 3 Sides $\times S.$ first Difference : Rectangle of the Sines of the 2^d and 3^d Differences ::

So Radius Square :

Tan. Square of half the included Angle.

For by Cor. 4. $S. \frac{z+b}{2} \times S. \frac{z-b}{2} : S. \frac{b+d}{2} \times S. \frac{b-d}{2}$

:: $(cc : ss ::) rr : \text{Tan. Square of half the Angle, by Prop. 1. I.}$

SCHO.

FIG.
46.

SCHOLIUM.

Any of these compound Proportions may easily be resolved into two simple ones. For Example in Cor.

4. say $S.a : r^2 :: S. \frac{b+d}{2} : \text{to the Sine of a fourth A.}$

Then $S.e : S.A : S. \frac{b-d}{2} : ss, \text{ required.}$ And the same may be done for the next Prop. and its Corollaries,

PROP. XLIII.

42

*In any spherical Triangle,
As Rectangle of the Sines of two Angles A, B :*

Radius Square ::

Vers. Sum of the including Angles—vers. Sup. 3^d Angle :

Vers. included Side ::

*And so vers. third Angle—vers. Sup. Sum of the including
Vers. included Side ::* [Angles :

And so Cos. diff. of the including Angles + Cos. 3^d Angle :

Vers. Sup. included Side ::

And so vers. Sup. 3^d Angle — vers. diff. including Angles :

Vers. Sup. included Side ::

And so vers. Sup. diff. including Angles—vers. 3^d Angle :

Vers. Sup. included Side, A B.

For let B be the greater Angle, A the lesser, A B the included Side, and put

$a=S.A, m=\text{Cos. } A, c=\text{Cos. } AB, v=\text{vers. } AB,$

$b=S.B, n=\text{Cos. } B, d=\text{Cos. } F, V=\text{vers. Sup. } A B,$

then (by Cor. 1. Pr. XXXIX.) $\frac{mnr+drr}{ab} = c = r - v,$

therefore $v = \frac{abr-mnr-drr}{ab}$; but (by Cor. 1. Pr.

5. I.) $\frac{mn-ab}{r} = \text{Cos. } A+B = (by Sch. 1. I.) r - \text{vers.}$

vers. $\overline{A+B} = \text{vers. Sup. } \overline{A+B} - r$, and $d = r - \text{vers. F} =$ FIG.
 vers. Sup. $\overline{F} - r$, whence $v = \frac{\text{vers. } \overline{A+B} - \text{vers. Sup. } \overline{F}}{ab}$ 42.
 $= \frac{\text{vers. } \overline{F} - \text{vers. Sup. } \overline{A+B}}{ab} rr.$

Again $V = r + c = \frac{abr + mnr + drr}{ab}$, But (by Cor. 1.

Pr. 6. I.) $\frac{ab+mn}{r} = \text{Cof. } \overline{B-A} = r - \text{vers. } \overline{B-A} = \text{vers.}$

Sup. $\overline{B-A} - r$. Therefore $V = \frac{\text{Cof. } \overline{B-A} + \text{Cof. } \overline{F}}{ab} xrr =$

$\frac{\text{vers. Sup. } \overline{F} - \text{vers. } \overline{B-A}}{ab} rr =$

$\frac{\text{vers. Sup. } \overline{B-A} - \text{vers. } \overline{F}}{ab} rr.$

Cor. 1. As *vers.* Sum of two Angles — *vers.* Diff. :

To twice *Radius* ::

Or as *Cof.* Difference of two Angles — *Cof.* Sum :

To twice *Radius* ::

So *vers.* Sum of the including Angles — *vers.* Sup.

Vers. included Side :: [third Angle :

So *Vers.* third Angle — *Vers.* Sup. Sum of the in-

vers. included Side :: [cluding Angles :

And so *Cof.* Diff. including Angles + *Cof.* 3^d Angle :

Vers. Sup. included Side ::

And so *vers.* Sup. 3^d Angle — *vers.* Diff. including Angles :

Vers. Sup. included Side ::

And so *vers.* Sup. Diff. incl. Angles — *vers.* 3^d Angle :

Vers. Sup. included Side, A B.

For (by Cor. 2. Pr. 3. I.) $S.A \times S.B =$

$\frac{\text{Cof. } \overline{B-A} - \text{Cof. } \overline{B+A}}{2} x r =$

$\frac{\text{vers. } \overline{B+A} - \text{vers. } \overline{B-A}}{2} x r.$ And all the rest fol-

lows from this Prop.

M

Cor.

FIG. Cor. 2. As Rectangle of the Sines of two Angles A, B:

42.

Radius Square ::

So S. $\frac{1}{2}$ Sum of the including $\angle$'s + $\frac{1}{2}$ Sup. $3^d \angle$ x

S. $\frac{1}{2}$ Sum including $\angle$'s - $\frac{1}{2}$ Sup. $3^d \angle$:

Sine Square of half the included Side ::

And so S. $\frac{1}{2}$ Sum of $3^d \angle$, and Sup. Sum incl. $\angle$'s x

S. $\frac{1}{2}$ their Diff. :

Sine Square of half the included Side ::

So S. $\frac{1}{2}$ Sup. $3^d \angle$ + $\frac{1}{2}$ Diff. incl. $\angle$'s x S. $\frac{1}{2}$ Sup. $3^d \angle$ -

$\frac{1}{2}$ Diff. incl. $\angle$'s :

Cof. Square of half the included Side ::

So S. $\frac{1}{2}$ Sup. Diff. incl. $\angle$'s + $\frac{1}{2}$ the $3^d \angle$ x S. $\frac{1}{2}$ Sup.

Diff. incl. $\angle$'s - $\frac{1}{2}$ the $3^d \angle$:

Cof. Square of half the included Side, A B.

For since the Diff. Cof. = Diff. versed Sines, therefore (by Cor. 1. Pr. 3. I.) $\frac{\text{vers. } B+A - \text{vers. Sup. } F}{2} \times r$

$$= S. \frac{B+A+\text{Sup. } F}{2} \times S. \frac{B+A-\text{Sup. } F}{2}$$

$$\text{And } \frac{\text{vers. } F - \text{vers. Sup. } B+A}{2} \times r =$$

$$S. \frac{F+\text{Sup. } B+A}{2} \times S. \frac{F-\text{Sup. } B+A}{2}$$

$$\text{And } \frac{\text{vers. Sup. } F - \text{vers. } B-A}{2} \times r =$$

$$S. \frac{\text{Sup. } F+B-A}{2} \times S. \frac{\text{Sup. } F+A-B}{2}$$

$$\text{And } \frac{\text{vers. Sup. } B-A - \text{vers. } F}{2} \times r =$$

$$S. \frac{\text{Sup. } B-A+F}{2} \times S. \frac{\text{Sup. } B-A-F}{2}$$

And (by Pr. 2. I.) $\text{vers. } A B \times \frac{1}{2} r = \text{Sine Square of } \frac{1}{2} A B$, and $\text{vers. Sup. } A B \times \frac{1}{2} r = \text{Cof. Square of } \frac{1}{2} A B$, then all the rest follows from the present Prop.

Cor.

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Cor. 3. Rectangle of the Sines of two Angles A, B :

FIG.

Radius Square ::

42.

Or Sum of the Sines $\times$ Diff. Sines, of $\frac{1}{2}$ Sum and $\frac{1}{2}$ Diff.

Radius Square ::

[Angles :

So Cos. $\frac{1}{2}$ Sum $\times$ Cos. $\frac{1}{2}$ Diff. of the 3^d $\angle$ and Sum incl.

[Angles :

Sine Square of half the included Side ::

So Cos. $\frac{1}{2}$ Sum $\times$ Cos. $\frac{1}{2}$ Diff. of 3^d $\angle$, and Diff. incl.

[Angles :

Cos. Square of half the included Side, A B.

For (by Cor. 2. Pr. 4. I.) $S.A \times S.B = S. \frac{B+A}{2} + S. \frac{B-A}{2}$

$\times S. \frac{B+A}{2} - S. \frac{B-A}{2}$; and $S. \frac{B+A+180-F}{2}$

$\times S. \frac{B+A-180+F}{2} = S. 90 + \frac{B+A-F}{2} \times S. \frac{B+A+F}{2}$

$-90 = \text{Cos.} \frac{F-A-B}{2} \times \text{Cos.} 180 - \frac{B+A+F}{2} = \text{Cos.}$

$\frac{F-A-B}{2} \times \text{Cos.} \frac{F+A+B}{2}$.

Again $S. \frac{180-F+B-A}{2} \times S. \frac{180-F+A-B}{2} =$

$S. 90 - \frac{F+A-B}{2} \times S. 90 - \frac{F+B-A}{2} = \text{Cos.}$

$\frac{F-B+A}{2} \times \text{Cos.} \frac{F+B-A}{2}$. The rest follows from

Cor. 2.

Cor. 4. From half the Sum of the three Angles, subtract each Angle at the Base separately, and then the third Angle. Then

As Rect. of the Cosines of the 2 first Differences :

Rect. of the Cos. 3^d Diff. and Cos. $\frac{1}{2}$ sum of 3 Angles ::

So Radius Square :

Tan. Square of half the included Side or Base, A B.

This follows from Cor. 3. because Cos. : Sine ::

Rad : Tangent. See Schol. Prop. XLII.

M 2

PROP.

PROP. XLIV.

48. If the vertical Angle F of a Triangle AFB be bisected, it will be
 As twice the Cot. of the bisecting Line FI:
 Sum of the Cotangents of the Sides AF, BF ::
 Radius :
 Cos. half the vertical Angle F.

Produce FA, FI, FB to Quadrants, and to the Pole F draw the great Circle LTR, and produce BA to intersect it in R. And let Tan. AS = b , Tan. IT = c , Tan. BL = d , S.RT = y , Cos. $\frac{1}{2}$ F = x . Then the Angles at S, T, L are right. And by Cor. I. Prop. XXVII. $c : b :: y : \frac{by}{c} = S.RS$. And $c : d :: y : \frac{dy}{c} = S.RL$. And since ST = TL; therefore (by Prop. 3. I.) $\frac{by+dy}{c} r = 2x \times y$, or $\frac{b+d}{c} \times r = 2x$, that is $2cx = \overline{b+d} \times r$.

Cor. As twice Radius :
 Tan. bisecting Arch ::
 Sum of the Cotan. Sides :
 Cos. half the vertical Angle F.

PROP. XLV.

48. If an Arch be drawn from the Vertex to the Middle of the Base of a Triangle; then the Sines of the vertical Angles are reciprocally as the Sines of the Sides, and directly as the Sines of the Angles at the Base.

For (by Cor. 1. Pr. XXIX.) S.AFI $\times$ S.AF = S.BFI $\times$ S.BF, because AI = BI. Therefore S.AFI : S.BFI :: S.BF : S.AF :: (by Pr. XXIX.) S.A : S.B.

PROP.

As one right Angle:

To the Angle intercepted between two great Circles ::

So the Area of a great Circle of the Sphere:

Lunular Area contain'd between these great Circles.

Let ABC , ADC be the two great Circles; BDF a great Circle whose Poles are A and C . Divide the Circle BDF into an infinite Number of equal Parts, thro' all which draw great Circles passing thro' A and C , which will divide the Surface of the Sphere into the same Number of equal Parts, similar and equal to one another; since their Bases (in the Circle BF), and Sides are all equal. Therefore as the Number of Parts in the whole Circumference, to the Number of Parts in BD , that is, as the whole Circumference, to the Arch BD :: so the Surface of the Sphere to the Area $ABCD A$; or as 4 right Angles: Angle BAD :: Surface of the Sphere: Area $ABCD A$. But (by Geometry) the Surface of the Sphere = 4 great Circles. Therefore as 4 right Angles: Angle BAD :: 4 great Circles: Area $ABCD A$; or as one right Angle : one great Circle :: Angle BAD : Area $ABCD A$.

Cor. As 4 right Angles:

Intercepted Angle BAD ::

So Surface of the Sphere:

Surface intercepted between these two great Circles ::

And so Solidity of the Sphere:

Solidity contain'd between these great Circles.

For the Solidities are as the Surfaces, because they may be resolv'd into Pyramids of equal height.

PRO P. XLVII.

In any Spherical Triangle, G ;

50.

As two right Angles:

Excess of the three Angles above two right Angles ::

Area of a great Circle of the Sphere:

Area of the Triangle.

M 3

Produce

FIG. Produce all the Sides to Semicircles, then the
 50. opposite Angles are equal (by Cor. 1. Prop. IV.); and
 (by Prop. IX) Triangle H = Triangle G, also half the
 Surface of the Sphere = $G + R + S + T$; let a, b, c be
 the three Angles, then (by Cor. Prop. XLVI.)

$$180 : a :: \frac{1}{2} \text{ Surface Sphere} : G + R$$

$$180 : b :: \frac{1}{2} \text{ Surface Sphere} : G + S$$

$$180 : c :: \frac{1}{2} \text{ Surface Sphere} : H + T \text{ or } G + T,$$

therefore $180 : a + b + c :: \frac{1}{2} \text{ Surface Sphere} : 3G + R + S + T$, and by Division, $180 : a + b + c - 180 :: \frac{1}{2} \text{ Surface Sphere} : 2G :: \frac{1}{4} \text{ Surface Sphere}$, or the Area of a great Circle : G.

Cor. 1. As 1 : 57.2957795 :: the 3 Angles — 2 right Angles : Area, in square Degrees.

For Area of a great Circle = $\frac{1}{2}$ Circumference $\times$ Radius = $180 \times \text{Radius}$, and Radius = 57.2957795 &c.

Cor. 2. As 180 : 3 Angles — 180 :: 3.1416 $\times$ Rad² : Area of the Triangle.

For 3.1416 $r r$ = Area of the Circle, whose Radius is r .

Cor. 3. In any spherical Polygon, put n = Number of Sides, $A = 180^\circ \times n - 2$. Then I say

As 2 right Angles or 180° :

Sum of all the Angles of the Polygon — A ::

So the Area of a great Circle of the Sphere :

Area of the Polygon.

Suppose the Polygon to be divided into as many Triangles as it has Sides, by Arches drawn from a Point within it. Then Since (by this Prop.) it is in any one Triangle, as 180 : a great Circle :: Sum of its Angles — 180 : its Area; therefore by Composition it will be, 180 : a great Circle :: Sum of all the Angles internal and external — $n \times 180$: Sum of all the Areas of the Triangles; but the Sum of the internal Angles = 360 or 2×180 , therefore 180 : great Circle :: Sum of all the Angles of the Polygon — A (or $n - 2 \times 180$) : Area of the Triangle. SECT.

SECT. IV.

The Solution of all the Cases of spherical Triangles.

By help of the Propositions delivered in the last Section, all the Cases of Spherical Triangles may be resolved; I shall give the Solution of each Case in particular, both in right angled Triangles and oblique Triangles, after I have shown how they may all be resolved in general by the foregoing Propositions. Every Spherical Triangle has 6 Parts, and any three being given the rest may be found.

The Solution of right angled Triangles.

All the Cases of right angled and quadrantal Triangles may be resolved by the 26 and 27 Propositions; or by Prop. XXVIII alone.

1. The Solution of any Case may be performed by the 26 or 27 Prop. one of the two; either immediately in the Triangle it self; or else in another right angled Triangle, form'd by producing all the Sides to Quadrants; wherein the Parts of the Triangle are either the same, or the Complements of those in the first. Thus, let ABF be the proposed Triangle, produce the Sides either thro' A or F , so that AC , AD , BE , and consequently CE may be Quadrants. Then you have two right angled Triangles ABF , EDF . In the Figure $ABCDEF$, mark what is given and its Complement with a Dash (/); and what is sought and its Complement with a Cypher (o). Then examine the two right Angled Triangles ABF , EDF ; and observe in which of them, *an Angle and the Hypotenuse, or an Angle and its opposite Side*, do

41.

FIG. not enter the Question, (for there are always two parts
41. out of the Question); and that Triangle will afford the Solution, by applying Prop. XXVII. in the former Case, and Prop. XXVI. in the latter. If any side be greater than a Quadrant, then a Quadrant must be cut off, and the same Rule apply'd.

Quadrantal Triangles may be resolv'd by the same Propositions, if you reduce them to right angled Triangles, by producing their oblique Sides to Quadrants, or cutting off a Quadrant if they be bigger. As in the Quadrantal Triangle AFE, produce AF, EF to Quadrants as AD, EB, and draw AB, ED round the Poles E, A, then you will have two right angled Triangles ABF, EDF, to be resolv'd by the former Rule. Take an Example or two for right angled Triangles.

EXAMPLE I.

51. Let the Hypothenuse AF and an Angle F be given, and the opposite Side AB required. I mark AF, F as given, and AB as sought, then since there is an Angle A, and its opposite Side FB out of the Question; therefore I need produce the Sides no further, for I have the Solution in the Triangle ABF, by Prop. XXVI. as $\text{Rad} : S. AF :: S. F : S. AB$ required.

EXAMPLE II.

41. Let the Hyp. AF, Angle A be given; to find the other Angle F, I produce the Sides thro' F, and mark AF, FD, A, DC, ED as given; and F as sought. Then I see that in the Triangle EFD, the Angle E and Hypothenuse EF are not mark'd, and therefore by Prop. XXVII. $S. FD : \text{Rad} :: \text{Tan. ED} : \text{Tan. Angle F}$, that is $\text{Cof. AF} : \text{Rad} :: \text{Cotan. A} : \text{Tan. F}$ required.

2. Right angled or quadrantal Triangles may also be resolv'd by Prop. XXVIII. Observing if middle

dle Part be sought, to begin with Radius; and if one FIG.
extream be sought, to begin with the other extream.

EXAMPLE I.

Given the Hyp. A F, and Angle F, to find the 51.
Side A B; what is given and sought being mark'd
as in the Figure. Then A B is middle Part, and A F,
F, extreams disjunct, therefore Rad: Cof. Comp. A F::
Cof. Comp. F : S. A B, that is Rad: S. A F:: S. F
: S. A B sought.

EXAMPLE II.

Given the Hyp. A F. and an Angle A, to find the 41.
Angle F. here A F is middle Part, and A, F extreams
conjunct. Therefore Tan. Comp. A : Rad:: S.
Comp. A F: Tan. Comp. F, or Cotan. A: Rad::
Cof. A F: Cotan. F.

EXAMPLE III.

In the quadrantal Triangle A F E, let Side A F, 41.
Angle A, be given; to find the Angle F. Here A F
is middle Part, and A and F are Extreams conjunct;
therefore.

Tan. E A F: Rad:: Cof. A F: Cotan. F.

Lastly the Affection of the Angle or Side sought
will be known by Prop. XX. and Prop. XXI and its
Corollaries; except it be ambiguous, and then it may
be either acute or obtuse.

The Solution of oblique spherical Triangles.

Oblique Triangles may be resolved several ways,
either by letting fall a Perpendicular, or without it.
In any Triangle mark what is given with a dash (/),
and what is required with a Cypher (o), to distinguish
them.

I. Oblique

FIG. 1. Oblique Triangles may be resolved by the Rules of right angled Triangles, together with the Corollaries of Prop. XXVIII. with Prop. XXIX, XXXII. XXXIII, by letting fall a Perpendicular, which either divides the Triangle into two right angled Triangles; or makes two right angled Triangles, by adding a right angled Triangle to it.

In letting fall the Perpendicular, let it fall from the End of a given Side, and opposite to a given Angle; and if the three Things given be adjoining to one another, let it also fall from the End of the Side required, or opposite to the Angle required; and then you will have enough given in one of the right angled Triangles to find any of its unknown Parts: observing to find such a Part (which must be either a Base or vertical Angle), as either of it self, or when added to or subtracted from some Part given, the Proportion to find the Thing sought may come under some of the Corollaries of Prop. XXVIII.

2. All the Cases of oblique Triangles may also be resolved by Prop. XXVII; XXIX, XXXII and XXXIII alone, by letting fall a Perpendicular according to the former Directions, when there is occasion. For the two last Cases will be resolved by Prop. XXXIII, and XXXII, and all the rest by Prop. XXVII and XXIX. and they are easily remembered, for three of them are the same with those in plain Trigonometry, for resolving the like Cases. Only there will be more Operations in the right angled Triangles, than if the Corollaries of Prop. XXVIII were made use of.

3. All the Cases of oblique Triangles (except the 1, 4, and 2 last) may also be resolv'd by Prop. XXVIII alone, by letting fall a Perpendicular dividing the whole into two right angled Triangles: In the first Triangle you have two Things given to find a Third, which must be either the Base or vertical Angle. In the second Triangle you have one Thing given, or at least you can find one Thing, after the first Operation

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tion in the first Triangle is over; and in the second FIG. Triangle there is also the Thing sought. Then in these two Triangles you have either both the Bases, or else both the vertical Angles. Therefore, from these, to find the Thing required, compare these three Things in the second Triangle, viz. the Perpendicular, the Thing given, and the thing sought, and find which is middle Part, and whether the other two be Extrems conjunct or disjunct, in order to know whether you must use Sines or Tangents. Do the same Thing in the first Triangle, with the three corresponding Parts; and (setting aside the Perpendicular, which is an Extream in both Triangles;) then the middle Part and its Extream in the first Triangle, will be proportional to the middle Part and its extream in the second Triangle.

For Example. Given FB, F and B; to find AF. In the first Triangle BFI, as $\text{Cotan. B} : \text{Rad} :: \text{Cof. FB} : \text{Cotan. BFI}$. then you have AFI; then in the Triangle AFI, the Angle AFI is middle Part, and AF, FI Extrems conjunct. Likewise in the Triangle BFI; BFI, is middle Part, and BF, FI, Extrems conjunct. Therefore $\text{Cof. BFI (middle Part)} : \text{Cof. AFI (middle Part)} :: \text{Cotan. BF (Extream conjunct)} : \text{Cotan. AF (Extream conjunct)}$.

53.

The Reason of this Operation will appear thus, let p =Perpendicular, m =middle Part, a =the other Extream, in the first Triangle. M =middle Part, A =Extream, in the second Triangle. Then by Prop.

XXVIII. $pa = mr$; and $pA = Mr$, therefore $\frac{p}{r} = \frac{m}{a} =$

$\frac{M}{A}$ or $m : a :: M : A$.

4. All oblique Triangles may also be resolved without letting fall a Perpendicular, and these five Propositions, the XXIX, XXXVI, XXXVII, XL, XLI, and their Corollaries will resolve them all. Thus, Case 1 and 4 are resolved by Prop. XXIX. Case 2 and 6 by Prop. XXIX and Cor. 1. Prop. XXXVII.
Case

FIG. 53. Case 3 and 5 by Prop. XXIX, and Cor. 1. Prop. XXXVII. Case 7 by Prop. XXXVI, and Cor. 1. Case 8, by Prop. XXXVI, and Cor. 1. and Prop. XXIX. Case 9 by Prop. XXXVII, and Cor. 1. Case 10 by Prop. XXXVII, and Cor. 1. and Prop. XXIX. Case 11 by Prop. XLII, or its Corollaries. Case 12 by Prop. XLIII, or its Corollaries.

Lastly the Affection of the Angle or Side sought may be gathered from Prop. XVII, and its Corollaries. And the falling of the Perpendicular, from Prop. XIX. and Cor. Prop. VIII.

Here follow the 16 Cases of right angled Triangles, and the 12 of oblique, In right angled Triangles, I shall give only the Solution by the Tables, omitting the Algebraic Solution; for that is very easily had by putting Letters instead of the Quantities, and transforming them by Sch. Prop. 1. I. if there is Occasion.

In oblique Triangles you have both; where note, that natural Sines, Tangents &c. must be used in the algebraic Way. I refer to the Cases in right angled Triangles, in the Determination of the Species of the Angle or Side sought, in the Method of letting fall a Perpendicular.



Right

Right angled spherical Triangles.

C A S E I.

Given the *Hypothenufe* AF, and an *Angle* A; to find the *side Adjacent* A B, 52.

Rad : Cos. Angle A :: Tan. Hyp. AF; Tan. adjacent Side A B.

If A, A F are of the same Affection, A B is less than a Quadrant; if of different Affection, more.

C A S E II.

Given the *Hyp.* AF, and *Angle* A; to find the opposite *Side* B F. 52.

Rad : S. Hyp. AF :: S. an Angle A : S. opposite Side BF.

If A be acute, B F is less than 90; if obtuse, more.

C A S E III.

Given the *Hyp.* A F, and an *Angle* A; to find the other *Angle* F. 52.

Rad : Cos. Hyp. A F :: Tan. an Angle A : Cotan. other Angle F.

When A F and A are of the same Affection, F is acute; if of different Affection, obtuse.

C A S E IV.

Given the *Hyp.* A F, and a *Leg* A B; to find the adjacent *Angle* A. 52.

Rad : Cot. Hyp. A F :: Tan. a Side A B : Cos. adjacent Angle A.

If A B, A F be of the same Affection A is acute; if of different Affection, obtuse.

C A S E

FIG.

CASE V.

52. Given the *Hyp.* AF and one *Side* AB; to find the opposite *Angle* F.
S. Hyp. AF : Rad :: *S. one Side* AB : *S. its op. Angle* F.
 If AB is less than 90, F is acute; if more than 90, obtuse.
-

CASE VI.

52. Given the *Hyp.* AF and a *Leg* AB; to find the other *Leg* BF.
Cof. one Leg AB : *Cof. Hyp.* AF :: Rad : *Cof. other Leg* BF.
 If AB, AF be of one Affection, BF is less than 90; if of different Affection, more.
-

CASE VII.

52. Given a *Side* AB, and its adjacent *Angle* A; to find the opposite *Side* F B.
Rad : *S. one Side* AB :: *Tan. an Angle* A : *Tan. opposite Side* F B.
 If A be acute, BF is less than 90; if obtuse, greater.
-

CASE VIII.

52. Given a *Side* AB and its adjacent *Angle* A; to find the opposite *Angle* F.
Rad : *S. an Angle* A :: *Cof. its adjacent Side* AB : *Cof. its opposite Angle* F.
 If AB is less than 90, F is acute; if more, obtuse.
-

CASE

SECT. IV. of TRIGONOMETRY.

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CASE IX.

FIG.

Given a *Side* AB, and *Angle* adjacent A ; to find the *Hyp.* AF. 52.

Cof. an Angle A : Rad :: Tan. adjacent Side AB : Tan. Hyp. AF.

If A and AB be of the same Affection, AF is less than 90 ; if different, more.

CASE X.

Given a *Side* BF, and opposite *Angle* A ; to find the *Side* adjacent AB. 52.

Rad : Cotan. an Angle A :: Tan. op. Side BF : S. adjacent Side AB.

Here AB is ambiguous, that is, it may be either lesser or greater than 90.

CASE XI.

Given a *Side* BF, and opposite *Angle* A ; to find the *Angle* adjacent, F. 52.

Cof. one Side BF : Cof. op. Angle A :: Rad : S. adjacent Angle F. ambiguous.

CASE XII.

Given a *Side* BF, and opp. *Angle* A ; to find the *Hyp.* AF. 52.

S. an Angle A : S. op. Side BF :: Rad : S. Hyp. AF, ambiguous.

CASE XIII.

Given the *Sides* AB, FB, to find an *Angle* A. 52.

S. one Side AB : Rad :: Tan. other Side BF : Tan. op. Angle A.

If BF is less than 90, A is acute ; if more, obtuse.

CASE

FIG.

CASE XIV.

52.

Given the *Sides* AB , FB ; to find the *Hyp.* AF .
Rad : *Cof. one Leg* FB :: *Cof. other Leg* AB : *Cof.*
Hyp AF .

If AB , FB are of one Affection, AF is less than a Quadrant; if of Different Affection, more.

CASE XV.

52.

Given the *Angles* A , F ; to find a *Side* BF .
S. one Angle F : *Rad* :: *Cof. other Angle* A : *Cof. op.*
Side BF .

If A be acute, BF is less than a Quadrant; if obtuse, it is greater.

CASE XVI.

52.

Given the *Angles* A , F ; to find the *Hyp.* AF .
Tan. one Angle F : *Cotan. other Angle* A :: *Rad* : *Cof.*
Hyp. AF .

If A , F be of the same Affection, AF is less than a Quadrant; if of different Affection, 'tis greater.



Oblique

Oblique Spherical Triangles.

C A S E I.

Given two *Sides* AF, BF, and an opposite *Angle* A; to find the other opposite *Angle* B.

54.
55.
56.

I. *Logarithmically, by the Table of artificial Sines, &c.*

S. one Side FB : S. op. Angle A :: S. other Side AF : S. its op. Angle B.

Then $\frac{A+B}{2}$ is of the same Affection as $\frac{AF+BF}{2}$.

And if $\frac{A+B}{2}$ be of the same Affection as $\frac{A+Sup. B}{2}$, then B is ambiguous.

II. *Algebraically, by the Table of natural Sines, &c.*

Let $d=S.AF$, $p=S.BF$, $a=S.A$; then $S.B = \frac{ad}{p}$.

C A S E II.

Given two *Sides* AF, BF, and an opposite *Angle* A; to find the included *Angle* F.

54.
55.
56.

I. *Logarithmically.*

By Case 3. right $\angle$ Triangles, $R : \text{Cof. AF} :: \text{Tan. A} : \text{Cotan. AFI}$.

And $\text{Cotan. AF} : \text{Cof. AFI} :: \text{Cotan. BF} : \text{Cof. BFI}$.

Then according as BF and Angle A are of the same or different Affection, BFI is acute or obtuse.

If $AFI > BFI$, then $AFI - BFI = F$ } if

If $AFI + BFI < 180$, then $AFI + BFI = F$ } both,
then F is ambiguous.

N

Or

FIG.

Or thus,

54.

Find the Angle B by Case I. then

55.

56.

$$S. \frac{AF \oslash BF}{2} : S. \frac{AF+BF}{2} :: Tan. \frac{B \oslash A}{2} : Cot. \frac{1}{2} F.$$

II. Algebraically.

47.

Let a, d, p be the Sines, and m, e, i the Cofines of A, AF, BF. And $b = \frac{ad}{p} = S.B, n = \text{Cof. B.}$ And
 $bb = bb - aa, = mm - nn, kk = 1 - bbpp = 1 - aadd.$
 Rad = 1.

$$\text{Then } S.F = \frac{bb}{bim - aen} = \frac{bim + aen}{kk}.$$

Also see Cor. 1, 2. Prop. XL.

CASE III.

54.

55.

56.

Given two Sides AF, BF, and an opposite Angle A; to find the third Side AB.

I. Logarithmically.

By Case I. right $\angle$ Triangles,

$$\text{Rad} : \text{Cof. A} :: \text{Tan. AF} : \text{Tan. AI}.$$

Then $\text{Cof. AF} : \text{Cof. AI} :: \text{Cof. BF} : \text{Cof. BI}.$

Then if BF and A are of one Affection, BI is less than a Quadrant; if of different Affection, more.

If $AI > BI, AI - BI = AB,$

If $AI + BI < 180, AI + BI = AB, \quad \left. \begin{array}{l} \text{If both, then} \\ \text{AB is ambiguous.} \end{array} \right\}$

Or thus.

* Find the Angle B by Case I. then

$$S. \frac{B \oslash A}{2} : S. \frac{B+A}{2} :: Tan. \frac{AF \oslash BF}{2} : Tan. \frac{1}{2} AB.$$

II. Algebra-

II. Algebraically.

Let a, d, p be the Sines, and m, e, i the Cosines of A , 47.
 $AF, BF. b = \frac{ad}{p} = S.B, n = \text{Cof. } B, \text{ and } ll = dd - pp$
 $= ii - ee, tt = 1 - bbbpp = 1 - aadd, \text{ Radius} = 1. \text{ Then}$
 $S.AB = \frac{ll}{mid - pen} = \frac{mid + pen}{tt}.$
 See also Cor. 1, 2. Prop. XLI.

CASE IV.

Given two Angles A, B , and an opposite Side BF ;
 to find the other opposite Side AF .

I. Logarithmically.

$S.A : S.BF :: S.B : S.AF.$

$\frac{AF+BF}{2}$ is of the same Affection as $\frac{A+B}{2}$. If
 $\frac{BF+AF}{2}$ be of the same Affection as $\frac{BF + \text{Sup. } AF}{2}$,
 then AF is ambiguous.

II. Algebraically.

Let $a = S.A, b = S.B, p = S.BF$, then $S.AF = \frac{pb}{a}$.

CASE V.

Given two Angles A, B , and an opposite Side BF ;
 to find the included Side AB .

I. Logarithmically.

By Case I. right $\angle$ Triangles, $\text{Rad} : \text{Tan. } BF ::$
 $\text{Cof. } B : \text{Tan. } BI.$
 Then $\text{Cotan. } B : \text{Cotan. } A :: S.BI : S.AI.$

FIG. If A, B be of the same Affection, $AI + BI = AB$.

54. If $BI + AI$ and $BI + \text{Sup. } AI$ be each < 180 , (AI and consequently) AB is ambiguous.

55. If A and B are of different Affection, $AI - BI = AB$. If both AI and $\text{Sup. } AI$ be $> BI$, then AI and AB are ambiguous.

Or thus.

47. Find AF by Case 4. then $S. \frac{B \oslash A}{2} : S. \frac{B+A}{2} ::$

$$\text{Tan. } \frac{AF \oslash BF}{2} : \text{Tan. } \frac{1}{2} AB.$$

II. Algebraically.

Let a, b, p be the Sines, m, n, i the Cosines of A, B, BF; $d = \frac{pb}{a} = S.AF$, $e = \text{Cof. } AF$, $ll = dd - pp = ii$

$-ee$, $tt = 1 - bbbp = 1 - aadd$, $\text{Rad} = 1$. Then

$$S.AB = \frac{ll}{\text{mid} - pen} = \frac{\text{mid} + pen}{tt}.$$

See also Cor. 1, 2, Prop. XLI.

CASE VI.

54. Given two Angles A and B, and an opposite Side

55. BF: to find the third Angle F.

56.

I. Logarithmically.

(By Case 3 right $\angle$'s) $\text{Rad} : \text{Tan} : B :: \text{Cof. } BF : \text{Cotan. } BFI$;

Then $\text{Cof. } B : S. BFI : \text{Cof. } A : S. AFI$.

If A and B are of the same Affection, $BFI + AFI = F$. If both $BFI + AFI$ and $BFI + \text{Sup. } AFI$ be < 180 , then (AFI and) F is ambiguous.

If A and B are of different Affection, $AFI - BFI = F$, if both AFI and $\text{Sup. } AFI > BFI$, then (AFI and) F is ambiguous.

Or

Or thus.

FIG.

Find A F by Case 4. then S. $\frac{A F \oslash B F}{2}$:

54.

S. $\frac{A F + B F}{2} :: \text{Tan. } \frac{B \oslash A}{2} : \text{Cotan. } \frac{1}{2} F.$

55.

56.

II. Algebraically.

Let a, b, p be the Sines; m, n, i the Cofines of A, B, 47.
 B F; $d = \frac{p b}{a} = S. A F$, $e = \text{Cof. } A F$. $bb = bb - aa = mm$

$-nn$, $kk = 1 - bbpp = 1 - aadd$, $\text{Rad} = 1$. Then

$$S.F = \frac{bb}{bim - aen} = \frac{bim + aen}{kk}$$

Also see Cor. 1, 2. Prop. XL.

C A S E VII.

Given two Sides A F, A B, and the included Angle A; to find an opposite Angle B.

54.

55.

56.

I. Logarithmically.

(By Case 1 right $\angle$'s) $\text{Rad} : \text{Cof. } A :: \text{Tan. } A F : \text{Tan. } A I$, and $A B \oslash A I = I B$.

Then S. A I : S. B I :: $\text{Cotan. } A : \text{Cotan. } B$.

According as A I is $<$ or $>$ than A B, B and A are of the same or different Affection.

Or thus.

$\text{Cof. } \frac{A B + A F}{2} : \text{Cof. } \frac{A B \oslash A F}{2} :: \text{Cotan. } \frac{1}{2} A :$

$\text{Tan. } \frac{F + B}{2}.$

And S. $\frac{A B + A F}{2} : S. \frac{A F \oslash A B}{2} :: \text{Cotan. } \frac{1}{2} A :$

$\text{Tan. } \frac{F \oslash B}{2}.$

Then

FIG.

54.

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56.

Then $\frac{F+B}{2} + \frac{F \oslash B}{2} = \text{Angle opposite to the great-}$
 er Side.

And $\frac{F+B}{2} - \frac{F \oslash B}{2} = \text{Angle op. to the lesser Side.}$

Note, $\frac{F+B}{2}$ is of the same Affection as $\frac{AF+AB}{2}$.

II. Algebraically.

47. Let s, a, d be the Sines, c, m, e the Cosines of AB , AF . $z = \text{Cotan. } AB$, $t = \text{Cotan. } AF$, $r = \text{Cotan. } A$, $\text{Rad} = 1$. Then

$$\text{Cotan. } B = \frac{se - dcm}{da} = \frac{s}{a}t - cr.$$

$$\text{Cotan. } F = \frac{dc - sem}{as} = \frac{d}{a}z - er.$$

C A S E VIII.

54. Given two Sides AF , AB , and the included
 55. Angle A ; to find the third Side FB .
 56.

I. Logarithmically.

(By Case 1 right $\angle$'s) $\text{Rad} : \text{Cos. } A :: \text{Tan. } AF : \text{Tan. } AI$, and $AB \oslash AI = BI$.

Then $\text{Cos. } AI : \text{Cos. } AF :: \text{Cos. } BI : \text{Cos. } BF$.

As BI and A are of the same or different Affection, BF is $\angle$ or $\oslash$ than a Quadrant.

Or thus

Find the $\angle B$ by Case 7, then $S. B : S. AF :: S. A : S. BF$, by Case 4.

II. Algebraically.

47. Let s, d be the Sines, and c, e the Cosines of AB , AF ; $m = \text{Cos. } A$, $v = \text{vers. } A$. $D = \text{Cos. } AB \oslash AF$. $\text{Rad} = 1$. Then $\text{Cos. } FB = s d m + c e = D - s d v$.

C A S E

CASE IX.

FIG.

Given two *Angles* A and F, and the included *Side* AF; to find an opposite *Side* FB.

54.

55.

56.

I. Logarithmically.

By *Case 3 right* $\angle$'s, *Rad*: *Tan.* A :: *Cof.* AF: *Cotan.* AFI. and $AFB \oslash AFI = BFI$; then,

Cof. AFI: *Cof.* BFI :: *Cotan.* AF: *Cotan.* BF.
as BFI and A are of the same or different Affection,
BF is $\angle$ or $\angle$ than a Quadrant.

Or thus.

$$\text{Cof. } \frac{F+A}{2} : \text{Cof. } \frac{F \oslash A}{2} :: \text{Tan. } \frac{1}{2} AF : \text{Tan. } \frac{AB+BFI}{2},$$

$$\text{And S. } \frac{F+A}{2} : \text{S. } \frac{F \oslash A}{2} :: \text{Tan. } \frac{1}{2} AF : \text{Tan. } \frac{AB \oslash BFI}{2},$$

Then $\frac{AB+BFI}{2} + \frac{AB \oslash BFI}{2} =$ Side opposite to the greater Angle.

And $\frac{AB+BFI}{2} - \frac{AB \oslash BFI}{2} =$ Side opp. to the lesser Angle.

Note, $\frac{AF+BFI}{2}$ is of the same Affection as $\frac{F+A}{2}$.

II. Algebraically.

Let a, d, f be the Sines, and m, e, g the Cosines of A, AF and F, $r = \text{Cotan. A}$, $t = \text{Cotan. AF}$, $y = \text{Cotan. F}$, Radius = 1, then

$$\text{Cotan. BF} = \frac{f r + g e}{d} = \frac{f r}{d} + g t;$$

$$\text{And Cotan. AB} = \frac{a y + m e}{d} = \frac{a y}{d} + m t.$$

CASE

FIG. I

CASE X.

54.

55.

56.

Two *Angles* A and F, and the included *Side* A F, being given; to find the third *Angle* B.

I. *Logarithmically.*

By *Case 3.* right $\angle$'s. $\text{Rad} : \text{Tan. A} :: \text{Cof. AF} : \text{Cotan. AFI}$, and $\text{AFB} \cup \text{AFI} = \text{BFI}$, then

$$\text{S.AFI} : \text{S.BFI} :: \text{Cof. A} : \text{Cof. B.}$$

As AFI is $<$ or $>$ than AFB, B and A are of the same or different Affection.

Otherwise.

Find FB by *Case 9*, then B by *Case 1*.

II. *Algebraically.*

47.

Let a, f be the *Sines*, and m, g the *Cosines* of A and F, $e = \text{Cof. AF}$, $\text{Rad} = 1$, then
 $\text{Cof. B} = afe - mg$. Also see Prop. XLIII.

CASE XI.

54.

55.

56.

Given the three *Sides*, to find an *Angle* A.

I. *Logarithmically.*

$$\text{Tan. } \frac{1}{2} AB : \text{Tan. } \frac{AF+FB}{2} :: \text{Tan. } \frac{AF \cup FB}{2} :$$

Tan. an Arch Q. If $\frac{1}{2} AB > Q$, the Perpendicular falls within; if less, without.

Then $\frac{1}{2} AB + Q =$ greater Segment AI or BR, and $\frac{1}{2} AB \cup Q =$ lesser Segment BI or AR. then

If $AF+FB < 180$, the Perpendicular falls nearest the lesser Side.

If $AF+FB > 180$, the Perpendicular falls nearest the greater Side.

Then by *Case 4.* right $\angle$'s. $\text{Rad} : \text{Tan. AI} :: \text{Cotan. AF} : \text{Cof. A.}$

Or

Sect. IV. of TRIGONOMETRY.

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Or thus,

FIG.

$$S. AB \times S. AF : Rad. Square :: S. \frac{BF+AB-AF}{2} \times \begin{matrix} 54. \\ 55. \\ 56. \end{matrix}$$

$$S. \frac{BF+AF-AB}{2} : S. Square \frac{1}{2} A.$$

$$Or S. AB \times S. AF : Rad. Square :: S. \frac{BF+AB+AF}{2}$$

$$\times S. \frac{BF+AB+AF}{2} - BF : Cos. Square \frac{1}{2} A.$$

See also Prop. XLII and Cor.

II. Algebraically.

Let d, s be the Sines, and e, c the Cofines of AF , 47-
 AB ; $i = Cos. FB$, $z = Cos. \frac{AB+AF}{2}$, $x =$
 $Cos. \frac{AB \cos AF}{2}$, Radius = 1, then $Cos. A = \frac{i-ec}{ds}$,
 and $vers. A = \frac{x-i}{x-z} \times 2$.

C A S E XII.

The three Angles being given; to find a Side AF .

I. Logarithmically.

As A, B are of the same or different Affection; the Perpendicular falls within or without.

$$Cot. \frac{A+B}{2} : Tan. \frac{A \cos B}{2} :: Tan. \frac{1}{2} F : Tan. \text{an Arch } Q.$$

Then $\frac{1}{2}F + Q =$ greater Angle at the Vertex AFI or BFR .

And $\frac{1}{2}F \cos Q =$ lesser Angle at the Vertex BFI or AFR .

Then if $A+B < 180$, the Perpendicular falls nearest the lesser Side or greater Angle.

If $A+B > 180$, the Perpendicular falls nearest the greater Side or lesser Angle.

Then by Case 16. right Angles, $Tan. AFI : Cotan. A :: Rad. : Cos. AF$.

O

Or

FIG.

Or thus.

54.

$$S.A \times S.F : \text{Rad. Square} :: \text{Cof. } \frac{B+F+A}{2} \times$$

55.

$$\text{Cof. } \frac{F+A-B}{2} : \text{Sine Square } \frac{1}{2} A.F.$$

56.

$$\text{Or } S.A \times S.F : \text{Rad. Square} :: \text{Cof. } \frac{B+F-A}{2} \times$$

$$\text{Cof. } \frac{B+A-F}{2} : \text{Cof. Square } \frac{1}{2} A.F.$$

Also see Prop. XLIII. and Cor.

II. Algebraically.

47.

Let a, f be the Sines, and m, g the Cosines of A and F , $n = \text{Cof. } B$, $z = \text{Cof. } F+A$, $x = \text{Cof. } F \cap A$, $\text{Rad} = 1$. Then

$$\text{Cof. } AF = \frac{mg+n}{af}, \text{ and } \text{vers. } AF = \frac{n+z}{z-x} \times 2.$$

S C H O L I U M.

Any of the Cases of right angled Triangles is resolved at one Operation; but in Obliques, all but the 1, 4, 11 and 12 require two Operations, but then they find two things for these two Operations, without a Perpendicular. I think it needless to give one Example in Numbers, the Directions in general for performing the arithmetical Work in Spherics, being the same as in plain Triangles.

F I N I S.

A D V E R T I S E M E N T.

THE Doctrine of Fluxions, not only explaining the Elements thereof, but also its Application and Use in the several Parts of Mathematics and natural Philosophy. By W. Emerson. Sold by W. Innys, in Pater-noster Row, London.

Where may be had, by the same Author,

The Projection of the SPHERE, orthographic, stereographic, and gnomonical.

